Bayesian Phylogenies Unplugged: Majority Consensus Trees with Wandering Taxa

September 13, 2011

Abstract

Probability distributions on phylogenetic tree topologies cannot be summarized by a mean and variance or by a confidence interval (a, b) because they are discrete and unordered. Researchers have therefore devised a number of visual summaries that represent a topology distribution in terms of a single topology that may contain multifurcations. One commonly used summary, known as the majority consensus tree, represents the "center" of a topology distribution by combining bi-partitions of leaf taxa (full splits) that individually have strong support into a single topology.

However, if a single taxon or clade wanders widely over the topology of the remaining taxa, the majority consensus tree may collapse to a star tree that contains no information, even if the topology of most taxa is fairly certain. This can cause substantial problems in interpreting the results of phylogenetic inference because the addition of new taxa can cause what looks like a large decrease in phylogenetic resolution. This situation can occur when (i) adding new taxa that are coded with missing data, (ii) when adding new taxa that lead to the creation of short branches by subdividing existing branches, and (iii) when the new taxa are connected to the remainder of taxa via a very long branch, as is the case for many outgroups. We also note that entire subtrees may wander, as opposed to just individual leaf taxa. In all these cases, the topology distribution may contain substantial structure although few splits are strongly supported. Instead, the structure of the topology distribution may be summarized in terms of supported splits on subsets of leaf taxa (called partial splits). However such structure is invisible to the majority consensus tree, which considers only splits on all leaf taxa (full splits).

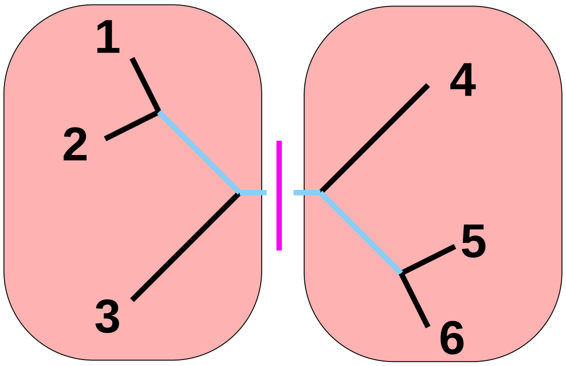
We present a new summary for topology distributions in order to assist researchers by revealing previously hidden structure in Bayesian posterior distributions and bootstrap frequency distributions. This improved summary can represent wandering clades as subtrees with several alternative attachment points, and can simultaneously represent the branches over which they wander, so that it is not necessary to prune the wandering clades from the tree. We term this new object a multiconnected (MC) tree, and note that it generalizes the traditional multifurcating (MF) tree. We show that a multiconnected tree naturally corresponds to a set of possibly partial splits induced by its branches, just as a multifurcating tree corresponds to a collection of full splits induced by its branches. We describe pairwise compatibility rules for partial splits that determine whether a set of partial splits forms a multiconnected tree. These rules ensure that there is at least one bifurcating tree that displays all the (possibly partial) splits in the collection, and that the collection does not imply any new quartets. This ensures that the multiconnected tree graph will not incidentally represent any unsupported splits, and therefore makes it a useful visual representation of supported partial splits. While checking the compatibility of a collection of splits is in general NP-complete if the splits may be partial, we note that for multiconnected trees a polynomial-time algorithm is possible. Finally, we describe an initial algorithm for finding supported partial splits in a collection of topologies.

1 Introduction

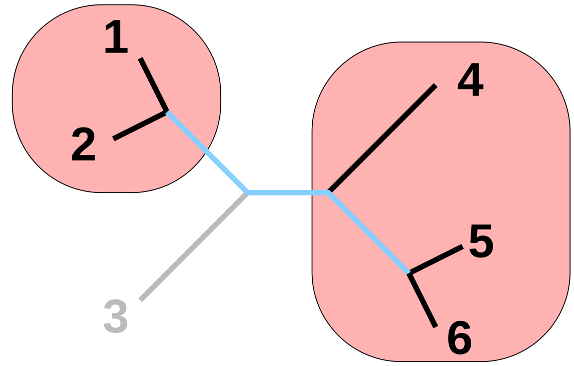
Probability distributions on phylogenetic tree topologies cannot be summarized by a mean and variance or by a confidence interval (a, b) because they are discrete and unordered. Visual summaries of topology distributions are therefore essential for a practical understanding of the evidence for and against phylogenetic hypotheses. Although it is possible to represent a confidence set in topology space as a list of trees, this representation does not have the compact representation, easy interpretation, and useful structure of an interval (a, b). To overcome this difficulty, researchers have devised a number of methods to summarize information from topology distributions in terms a single topology that may contain multifurcations. One commonly used summary, known as the majority consensus tree, represents the "center" of a topology distribution by combining bi-partitions of leaf taxa (full splits) that individually have strong support into a single topology. However, if the tree is subdivided into short branches, or if a clade wanders widely over the topology of the remaining branches, there may be no full splits that are highly supported. The majority consensus tree may then collapse to a star tree that contains no information, even when the topology distribution contains substantial structure, such as having many supported quartets. The inability of the majority consensus to represent branches near uncertain attachment points may lead to incorrect claims that few topological relationships can be resolved by the data. This difficulty may also discourage researchers from including data about additional taxa in an analysis, because of the misperception that the expanded data set contains less information than the original, smaller data set.

To remedy this, we extend the majority consensus tree in two ways. First, we extend the majority consensus tree to include supported splits that partition only some of the leaf taxa. Such splits are known as partial splits, while splits that partition all the leaf taxa are known as full splits (Figure 1). Partial splits represent much of the common information between topologies that differ in the placement of a wandering taxon. Partial splits may achieve high support despite the presence of wandering clades by partitioning only non-wandering taxa. Second, we introduce a generalization of multifurcating trees that we call a multiconnected tree graph (Figures 2d, 3a). A multiconnected tree graph allows branches of the tree to specify a number of alternative attachment points instead of attaching in only one place as usual. This enables multiconnected trees to graphically represent uncertainty about attachment points of wandering clades. Such graphs permit our new summary to reveal hidden structure in topology distributions by graphically representing both wandering clades and the branches over which they wander (Figure 5). In contrast, alternative methods such as the majority consensus tree cannot display partial splits; to reveal hidden branches they must prune wandering clades from the tree so that the partial splits become full splits on the reduced taxon set (Wilkinson 1994).

Surprisingly, the handling of uncertain attachment points via partial splits and via multiconnected tree graphs may be completely unified. We show that multiconnected tree graphs can be represented in terms of the partial splits that are induced by each branch (Figure 3b), and we show how to construct multiconnected tree graphs from collections of partial splits. We may therefore construct a multiconnected tree graph from a set of partial splits that is supported under a probability distribution on tree topologies. However, not every compatible collection of partial splits corresponds to a multiconnected tree graph. We therefore provide simple rules to determine which collections of partial splits are not only compatible, but can also be represented as a multiconnected tree graph. We describe a method of discarding additional unrepresentable splits to obtain a representable split set and obtain a multiconnected tree graph as a summary. We term this summary for topology distributions the extended majority consensus (EMC).



(a) Full split: 123|456



(b) Partial split: 12 |456

Figure 1: Full splits and partial splits. (a) A full split is a bi-partition of all leaf taxa. Each branch of a tree induces a full split on that tree's leaf taxa when cut. (b) A partial split is a bi-partition of only some of the leaf taxa. Note that whether a split is full or partial depends on the set of leaf taxa being considered, so that it is possible for a split to be a full split with respect to one set of leaf taxa, but a partial split with respect to a larger set of leaf taxa.

Topology distributions and consensus methods In the Bayesian paradigm (on which we focus here), the posterior probability distribution on topologies represents the evidence for and against phylogenetic hypotheses. However, topology distributions may arise from a variety of different inference methods. When maximum likelihood or maximum parsimony is employed to construct an estimate of the topology, bootstrap proportions are often employed to assess confidence in topologies or clades. We note that bootstrap proportions are not to be interpreted as the probability that a tree is correct; the bootstrap proportion for a clade should not be expected to be identical to its posterior probability (Huelsenbeck and Rannala 2004). Nonetheless, bootstrap proportions lead to a probability measure on topologies in the mathematical sense that the probability summed over all topologies is 1. Finally, consensus trees have been used to summarize common information in finite sets of trees, such as the set of most parsimonious trees from a maximum parsimony analysis. Such a set can be considered to represent a probability distribution by assigning equal probability to each topology in the set.

Systematists have described many different consensus tree methods to summarize common information in collections of tree topologies (Bryant 2003). Several of these methods, such as the strict consensus tree, and the maximum agreement subtree (MAST), only represent information that is common to 100% of input trees. These methods are appropriate for summarizing common information in a set of topologies, but are not suitable for Bayesian posterior probability distributions. This is because the posterior probability of a topology or a non-trivial clade will always be less than 1.0 (unless alternatives are forbidden *a priori*), and so the summary tree will not contain any internal branches. Posterior tree samples generated by Markov chain Monte Carlo (MCMC) may share some information by strict agreement, but this agreement must disappear as the sample size approaches infinity and stochastic error decreases.

The majority consensus tree and generalizations The majority consensus tree is appropriate for summarizing Bayesian posterior distributions and bootstrap fractions because it can represent information that is not shared by all input trees. Like many other methods, the majority consensus tree exploits the fact that each branch splits the leaf taxa into two groups when cut, and that each tree can be uniquely decomposed and represented in terms of these bi-partitions, or "splits". The

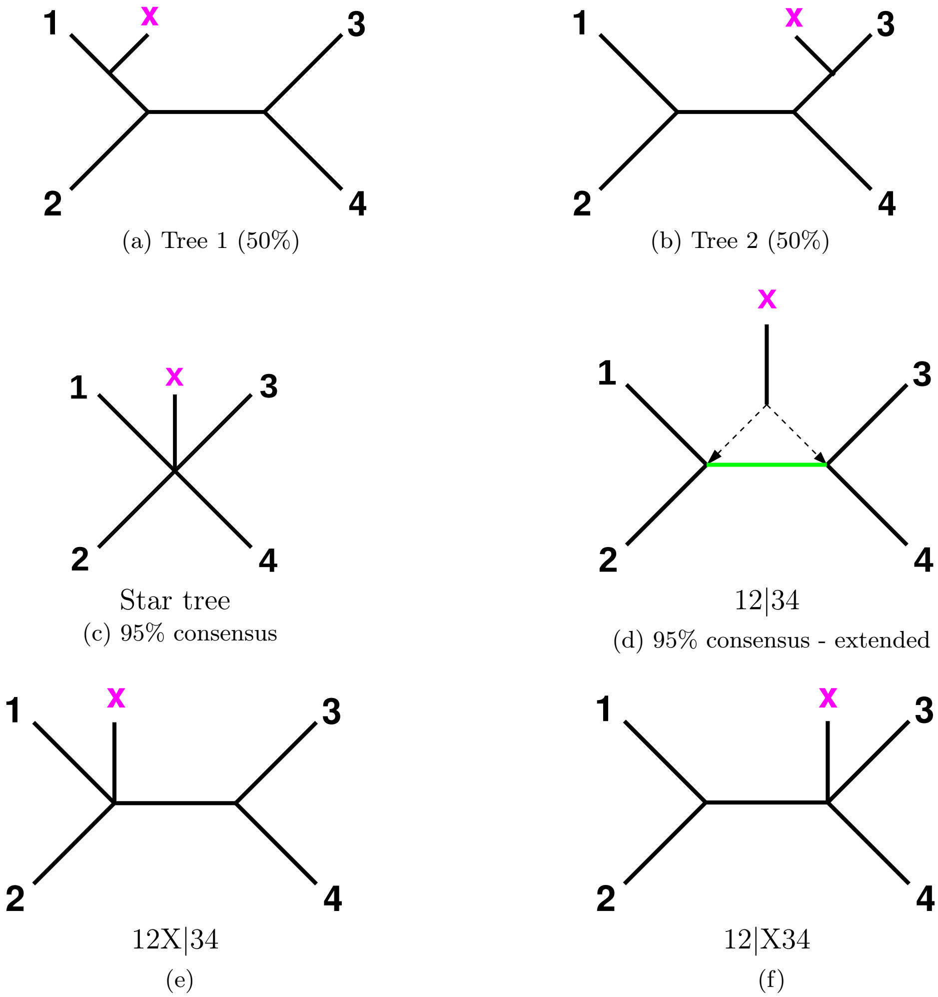
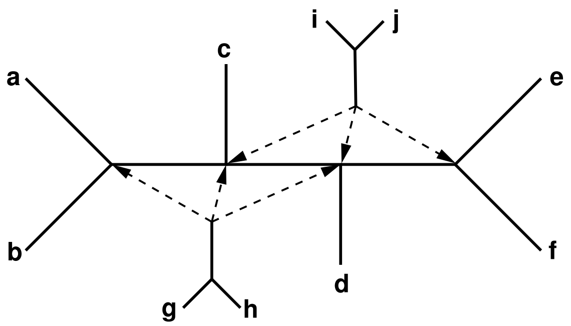


Figure 2: Under the probability distribution that places equal weight on the trees depicted in (a) and (b), only the position of X is uncertain. However, the 95% majority consensus tree (c) for this distribution has no internal branches and fails to display all the common information in (a) and (b). In contrast, the 95% extended majority consensus (d) has an internal branch that induces the partial split 12|34. Because the taxon X is not involved in the split, it may attach on either side of the internal branch. To determine the set of bifurcating trees that are compatible with (d), we first note that it embeds two multifurcating trees (e) and (f). Second, we note that each of the multifurcating trees (e) and (f) is compatible with 3 bifurcating trees, one of which is consistent with both (e) and (f). Therefore 5 bifurcating trees are compatible with the multiconnected tree (d). Therefore, the multiconnected tree (d) represents a set of trees in which the wandering taxon X may attach first to any of its attachment points (producing a multifurcating tree), and then to any branch that is adjacent to these attachment points (resolving the multifurcations to produce a bifurcating tree).



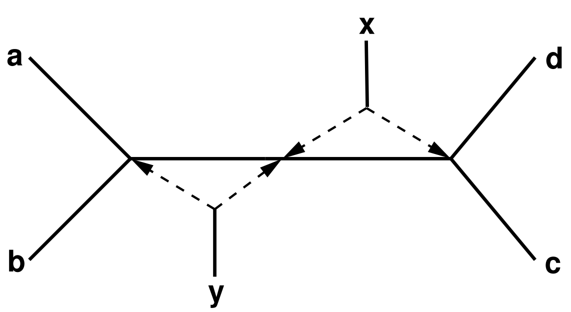
(a) A multiconnected tree graph.

Partial Splits
ab|cdijef
abc|def
abghcd|ef

Full Splits
gh|abcdefij
ij|abcdefgh

(b) Informative splits

Figure 3: A multiconnected tree graph and its informative splits. (a) The multiconnected tree graph contains two types of edges. Type 1 edges (solid lines) are undirected and correspond to splits. Type 2 edges (dashed lines with arrows) are directed and point to alternative attachment points for Type 1 edges. If a subtree can attach to both endpoints of a Type 1 edge, then we say that this subtree and all its leaf taxa "wander across" the edge. Clades such as $\{g, h\}$ may wander jointly; it is not just individual leaf taxa that wander. It is also possible for two different subtrees to wander across the same branch. (b) Each Type 1 edge corresponds to a split of all taxa that do not wander across it. Informative splits correspond to internal branches; here we ignore splits that correspond to leaf branches.

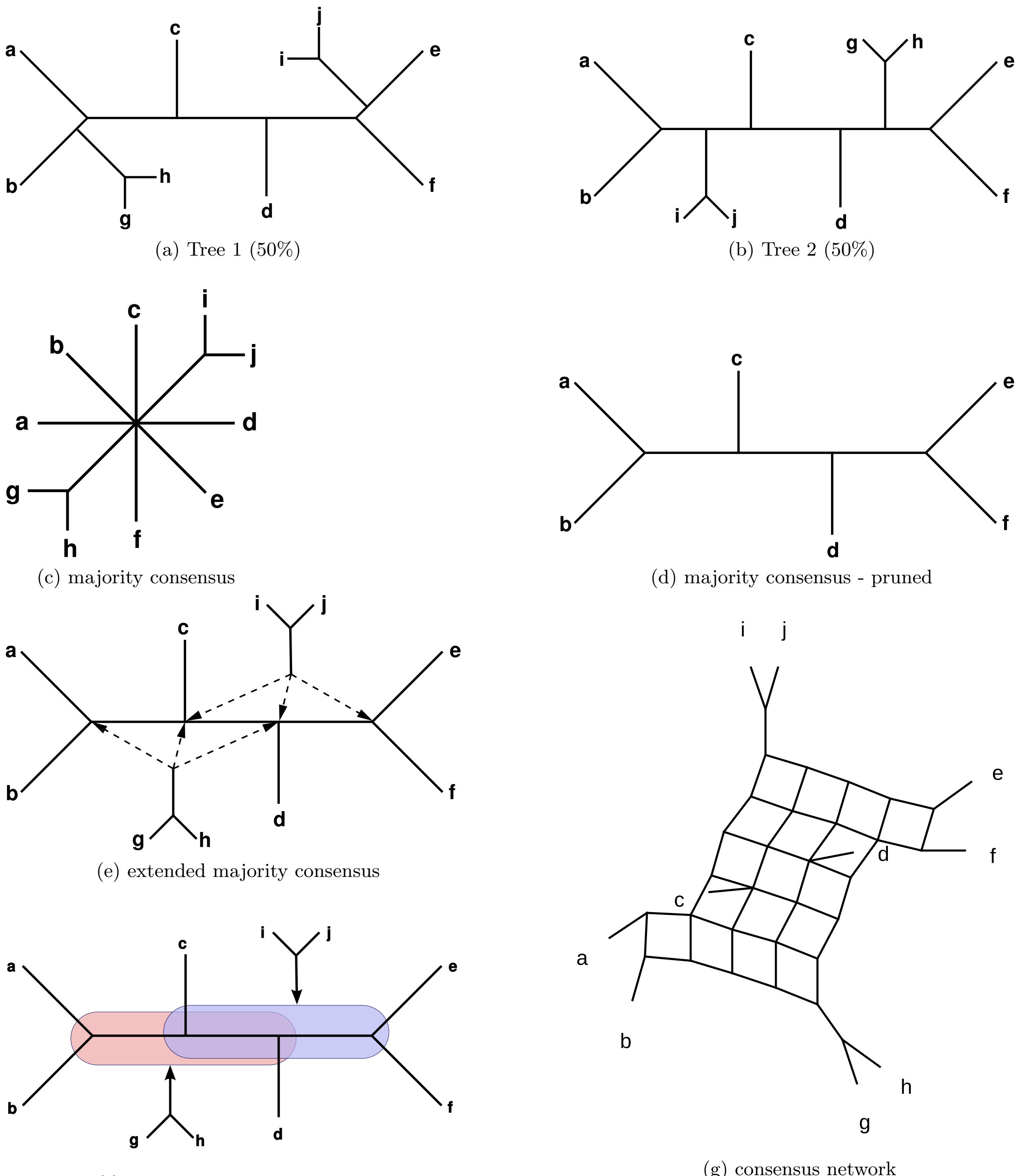


(a) A multiconnected tree graph with a special node.

Partial Splits
ab|xcd
aby|cd

(b) Informative splits.

Figure 4: A multiconnected tree graph can contain special nodes that are connected to only two edges. This multiconnected tree graph represents ambiguity in the order of adjacent taxa x and y. (a) The node to which x and y may both attach is a special node. If x and y are both attached to this node, then the graph becomes a multifurcating tree that can be further resolved to place either x or y on the left. Representing such ambiguity is important when there are many short branches on the tree. If neither x nor y attach to this node, then the node must be removed from the graph. If this is not done, then two adjacent branches will induce the same split. (b) Informative splits that correspond to the internal branches of this multiconnected tree graph.



(a) Tree 1 (50%)

(b) Tree 2 (50%)

(c) majority consensus

(d) majority consensus - pruned

(e) extended majority consensus

(f) extended majority consensus

(g) consensus network

Figure 5: Wandering taxa obscure branches in the majority consensus. Consider a topology distribution that places equal weight on the topologies in (a) and (b). (c) The majority consensus tree displays little information. (d) If we remove the taxa g, h, i, and j, then three new internal branches are revealed. (e) The extended majority consensus can represent these three branches along with the wandering clades $\{g, h\}$ and $\{i, j\}$. (f) The cloud representation for the same multiconnected tree graph. (g) The 30% consensus network is able to display incompatible splits in the same graph. This network displays much of the same information, but does not show the high degree of support for partial splits.

majority consensus tree is constructed by first collecting all full splits with posterior probability greater than one half, and then constructing a multifurcating tree in which each branch corresponds to exactly one of the collected full splits.

The majority consensus can also be generalized to represent full splits at a desired level l of support. We follow F.R. McMorris and Neumann (1983) in defining the M_l tree as the unique multifurcating tree which contains all full splits that are supported with a probability greater than l. This M_l tree is guaranteed to exist if l is 0.5 or larger (Margush and McMorris 1981). However, full splits that are supported at a level $l < 0.5$ may not be compatible with each other, and so there may be no tree that displays all such full splits. By adjusting l, the researcher may construct a several different M_l consensus trees that contain full splits with various levels of support. Using this notation, the $M_{0.5}$ tree is then the standard majority consensus tree. We abuse the notation slightly by using $M_{1.0}$ to refer to the strict consensus tree, even though the full splits in this tree have support that is only equal to 1.0 instead of being greater than 1.0.

1.1 Problems with the majority consensus tree

One substantial drawback of the majority consensus tree is its sensitivity to taxon sampling. As data from more taxa are used to compute a Bayesian posterior probability distribution or a bootstrap frequency distribution, the majority consensus tree for that distribution may become less resolved, in the sense that it contains fewer internal branches. Conversely, removing certain taxa from an analysis may lead to a majority consensus tree that contains more internal branches. Unless these taxa are removed, the majority consensus tree may fail to represent supported splits.

The conventional approach to this drawback is to attempt to increase the number of internal branches in the majority consensus tree by "pruning" leaf taxa from the tree. Ideally, such pruning would involve removing taxa only from the topology distribution (or from the sampled trees that represent this distribution); researchers should not remove these leaves and their associated observations from the data set, in order to avoid discarding any information in the original data set about topology of the remaining leaves. However, in practice, researchers often censor their data sets for fear that the resulting majority consensus tree will be less resolved (**cite**).

There are two main causes for the sensitivity of the majority consensus to taxon sampling. These causes are described in greater detail below, but we introduce them here. First, sensitivity to taxon sampling may be caused by a few taxa or clades that may attach at many different places to the topology of the other taxa. These taxa are known as wandering taxa, rogue taxa, or wildcard taxa. This problem typically occurs when a taxon or clade contains missing data or is connected to the remaining taxa by a very long branch, because these conditions reduce the about of information at the attachment point about the ancestral sequence. Addition of a wandering taxon that attaches on either side of an originally supported branch apportions the original support for the branch between these two alternatives. Thus, a highly supported full split on the original taxon set may correspond to two new full splits on the new taxon set that differ only in the placement of the wandering taxon; these two new full splits may have a substantially lower probability. Conversely, removing a wandering clade merges these two alternatives back together, thus leading to a single branch with increased support.

Second, sensitivity to taxon sampling may be caused by short, adjacent branches in the true tree. If a clade attaches to the true tree on a short branch, then adding that clade may obscure the short branch in the majority consensus. This phenomenon does not depend on the amount of information about the ancestral sequence at the attachment point and so is not influenced by whether the new clade has missing data or is connected by a long branch. Instead, adding a clade to the tree subdivides the short branch into two shorter branches. In general, shorter branches are more weakly supported by the data since fewer character changes occur on them. Conversely, removing a clade from the tree merges two subdivided branches into one longer branch which will be more strongly supported. In this scenario, many branches may be interchanged with their near neighbors since the branches separating them are short, but they may not be interchanged with more distant taxa. This scenario differs from the wandering taxon scenario because it is not possible to designate any specific taxon or clade as the culprit for the lack of resolution in the consensus tree. Instead, many alternative taxa may be pruned to increase the length of internal branches (Figure 6). Finally, we note that as we progress away from the leaf nodes which contain observed data and towards the interior of large trees, the attempt to cleanly separate the causes of wandering clades into these two categories may break down.

1.2 Partial splits and multiconnected trees

An important point of this paper is that in both the short branch scenario and the wandering taxon scenario, the information that is missing from the majority consensus tree is captured by partial splits. This partial split information may then be displayed in the extended majority consensus using multiconnected trees. Because the majority consensus tree only represents supported full splits, it cannot represent this information without pruning leaf taxa.

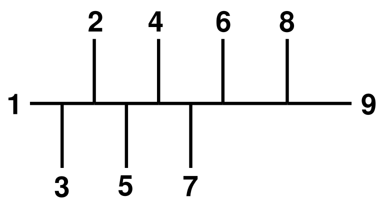
The wandering taxon problem When a new clade with taxa X is added to a tree, it may have a substantial probability of attaching on each side of a branch with full split $A_1|A_2$. In such a case, we say that X wanders over the branch. When X attaches on one side of the branch, the full split $A_1 \cup X|A_2$ is induced, but when X attaches on the other side of the branch, the full split $A_1|X \cup A_2$ is induced instead; when X attaches in the middle of the branch, both splits are induced. Because each of these two full splits partition an increased number of taxa, their probability must be lower than the split $A_1|A_2$ for a given data set. In fact, the split $A_1|A_2$ is induced in all three cases, and the cases are non-overlapping, so the probability of $A_1|A_2$ is equal to the sum of the probabilities of these three cases. Therefore, in the worst case where X attaches with equal probability on each side of the branch and rarely attaches on the branch itself, we may have that

$$\Pr(A_1 \cup X|A_2) = \Pr(A_1|X \cup A_2) \quad = \quad \Pr(A_1|A_2)/2.$$

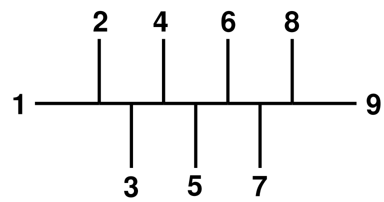
When this occurs, neither $A_1 \cup X|A_2$ nor $A_1|X \cup A_2$ will be part of the majority consensus tree, even if $\Pr(A_1|A_2)$ is as high as 1.0. In such a situation it would be advantageous for a graphical consensus to represent the split $A_1|A_2$, even though this split is a partial split on the enlarged taxon set.

Figure 5 illustrates this scenario by placing an equal probability of 50% on two different topologies (Figures 5a,b). The clades {g,h} and {i,j} may attach at widely different points on the topology of the remaining taxa, leading to a 95% consensus tree with few internal branches (Figure 5c). However, when these two wandering clades are removed, the 95% consensus reveals substantial structure in the topology distribution for the remaining taxa (Figure 5d).

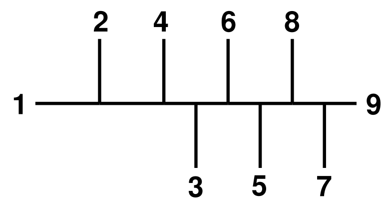
Consider the branch with split ab|cdef in figure 5d and the effect of adding the clade {g,h} to the tree. While the split ab|cdef has probability 100%, this split will become a partial split after the addition of {g,h}. We will then be forced to consider the two full splits abgh|cdef and ab|ghcdef. These full splits are identical except for their placement of the {g,h} clade. Each split has a probability of only 50%, and therefore both splits will be excluded from a 95% consensus tree. As a result, the addition of the clade {g,h} to the tree will result in the removal of the highly supported branch ab|cdef because it is a partial split, but will not replace it will either of the full splits abgh|cdef or ab|ghcdef because their probability is too low.



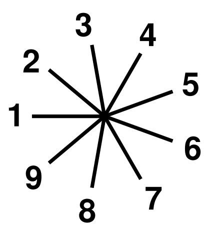
(a) Tree 1 (33%)



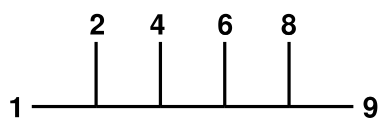
(b) Tree 2 (33%)



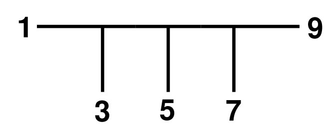
(c) Tree 3 (33%)



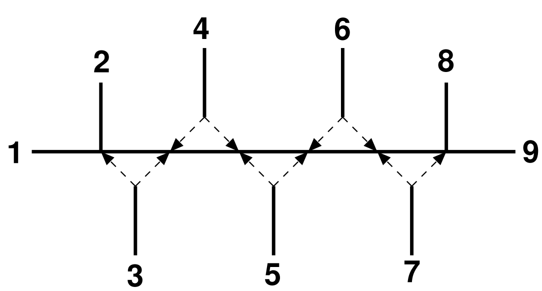
(d) 95% consensus



(e) 95% consensus -3,-5,-7



(f) 95% consensus, -2,-4,-6,-8



(g) 95% extended consensus

Figure 6: The short branch problem. Consider a topology distribution that places equal weight on the topologies in (a), (b) and (c). In this distribution, each branch has substantial probability of being interchanged with near neighbors, but lower probability of being exchanged with far neighbors, as would be the case of the number of mutations on each internal branch was small. More realistic distributions would not have even and odd taxa moving in lock step, but would require much more than 3 fundamental trees. (d) No internal branches are supported at the 95% level, so that the 95% consensus is the star tree. (e) However, pruning taxa 3, 5, and 7 leads to 3 longer branches that are supported at the 95% level. (f) Likewise, pruning taxa 2, 4, 6, and 8 leads to 2 longer branches that are supported at the 95% level. (g) The extended majority consensus is able to display all 5 branches without pruning any taxa, because it can represent uncertainty in attachment points.

However, we note that removing the clades {g,h} and {i,j} from the tree has a cost as well as a benefit. Pruning these clades reveals three new internal branches (figure 5d) but removes the two internal branches that group {g,h} and {i,j}. Because multifurcating trees cannot display partial splits, it is not possible to display all five supported splits in one tree. Instead, one must display both 5c and 5d, with different leaves pruned, in order to include all 5 supported branches. Thus, pruning clades involves an inherent tension between the goal of revealing the hidden branches over which the clades wander, and displaying the internal structure of the clades themselves. In constrast, the use of multiconnected trees instead of multifurcating trees allows the 95% extended consensus to display all 5 supported branches in one figure, as well as indicating the range over which the rogue clades wander.

Adjacent short branches When a new clade with taxa X is added to a short branch with full split $A_1|A_2$, it subdivides this branch into two adjacent shorter branches that have full splits $A_1 \cup X|A_2$ and $A_1|X \cup A_2$. Because the subdivided branches are shorter, they will each be supported by a decreased number of character changes. In contrast, the original split $A_1|A_2$ corresponds to both subdivided branches and is supported by a larger number of character changes. Thus, as in the previous scenario, the majority consensus may be sensitive to taxon sampling, because the addition of the new clade with taxa X may result in the loss of the original split $A_1|A_2$ from the majority consensus without replacing it with either $A_1 \cup X|A_2$ or $A_1|X \cup A_2$. However, when the branch with split $A_1|A_2$ is short, this sensitivity may occur even when the new clade is not connected by a long branch and does not have missing data. Therefore, the new clade may not wander widely over the tree, but may wander a short distance from its true attachment point. If there are many short branches within this short distance, the new clade may wander over several of them.

When there are many adjacent short branches, we may expect that many clades may interchange with their near neighbors across these short branches. In such a situation, it does not make sense to label any of these interchanging clades as being "rogue" clades. This is, first, because the problem is caused by the shortness of the branches across which they are interchanging, and not because of any feature in the clades themselves. Second, there is not a unique clade that can be removed to restore the missing branches. Instead, any of the interchanging clades may be removed, thereby merging adjacent short branches into a longer branch that may be supported in the majority consensus.

We illustrate this alternative scenario by considering a concrete example in which leaf taxa have uncertain attachment points, and may be interchanged with their near neighbors. Figure 6 depicts a simple topology distribution that places equal support of 33.3% on each of three topologies (figure 6). We note that more realistic distributions (as in Figure 13) would not have even and odd taxa moving in lock step, but we choose not to illustrate such a distribution here because it would require much more than 3 fundamental trees. The 95% consensus tree contains no internal branches (figure 6d) because each leaf branch may attach on either side of its nearest neighbor. However, pruning every other leaf branch merges adjacent short branches to form long branches that have high enough support to be included in the 95% consensus tree; the resulting tree contains either 3 (Figure 6e) or 2 (Figure 6f) internal branches, depending on which leaves are pruned.

Consider the possiblity of adding leaf taxon 5 to the pruned 95% consensus tree illustrated in figure 6e. This tree contains the supported split 124|689. However, if the leaf taxon 5 were added, then the split 124|689 would no longer be a full split, and could not be added to the 95% consensus tree. Instead, we would be forced to consider the two splits 1245|689 and 124|5689 which each have a probability of 66%. These splits are equivalent to each other except for the position of taxon 5. Therefore, the addition of taxon 5 would remove an internal branch from the 95% consensus tree. Thus, the ability to display the partial split 124|689 would be advantageous.

The partial split 1234|6789 would be even more advantageous to display, because it partitions more leaf taxa. This split is displayed in the 95% extended majority consensus tree (figure 6g). Thus, the extended consensus tree is able to prune taxon 5 (and other leaf taxa) locally instead of globally.

Other approaches to wandering taxa: pruning- and subtree-based methods Other researchers have also addressed the problem of wandering taxa. Wilkinson (1994) introduced the Reduced Cladistic Consensus (RCC) in order to display partial splits. Wilkinson referred to partial splits as "n-taxon statements", indicating that they partition some number n of the leaf taxa, but not necessarily all of them. Because the RCC uses multifurcating trees to display splits, partial splits must be displayed by pruning taxa. Therefore the RCC summarizes topology distributions in terms of a large collection of multifurcating trees, each of which is a majority consensus tree or strict consensus tree with different leaves pruned. The goal of this method is to find the smallest number of different pruned leaf sets (and thus different trees) that must be displayed in order to represent all partial splits in a set of supported partial splits. This method broke new ground by making use of of partial splits. However, it can be cumbersome to use because of the large number of different trees that must be examined, each of which contains mostly the same information.

Cranston and Rannala (2007) recently introduced a method for pruning leaf taxa in order to find a single bifurcating topology that obtains a high probability. Given a number k of leaf taxa that may be pruned, this method determines the set of leaf taxa to prune that yields the highest probability topology. Because of the large number of such sets, the method uses a random walk algorithm similar to MCMC in order to maximize the probability of the resulting tree. This method is similar to the MAST consensus, but differs in three ways. First, it does not require strict agreement between all trees. Second, it yields bifurcating trees instead of multifurcating trees. Third, it yields a different result for each value of k, whereas the MAST method finds the smallest value of k that yields strict agreement between input trees. This method is also different than the majority consensus tree and related methods, because it considers the probability of the splits in a tree jointly rather than singly.

We note that Cranston and Rannala's idea could be extended to seek multifurcating trees instead of just bifurcating trees. The use of multifurcating trees would be a substantial improvement, because in cases of unresolved polytomies deep in the tree they would allow removing as little as one internal branch, instead of removing a large number of leaf taxa. However, implementing such a method would require the researcher to explicitly specify how to balance the relative penalties for removing internal branches and removing leaf taxa.

Other approaches to phylogenetic uncertainty: beyond multifurcating trees Several new consensus methods have been proposed that move beyond multifurcating trees to summarize topology distributions. For there are several different methods for using networks to represent phylogenetic information (Huson and Bryant 2006). However, some of these methods represent collections of different gene trees that result from hybridization, recombination, or other biological processes; in this paper we do not consider such networks and focus instead of methods that represent phylogenetic uncertainty. Holland et al. (2005) discuss using split networks instead of trees to represent collections of incompatible full splits that are supported in a topology distribution. This approach allows them to represent full splits with probabilities as low as 0.1 in the same figure, instead of the usual cutoff of 0.5. Bonnard et al. (2006) suggest using a small number of separate trees so that incompatible full splits can be represented in different trees. They show how to heuristically solve a graph-coloring problem in order to find a small number of separate trees that together represent a collection of supported splits that are not all compatible, although the splits in each tree must be compatible. They demonstrate that full splits with probabilities as low as 0.1 can usually be represented in a collection of only 4 trees.

Both of these methods attempt to improve the representation of topology distributions by displaying incompatible full splits with probabilities less than 0.5. This can be quite useful when a standard majority consensus tree has an internal node with a high degree that allows many possible topologies if only a small number of topologies occur with any substantial probability. In such a case, the two methods mentioned above illustrate which conflicting alternatives actually occur by displaying the incompatible alternative splits with probability between 0.1 and 0.5. Thus both of these methods may add valuable information to the majority consensus tree.

However, these methods do not specifically address the problem of wandering taxa. Neither of the two graphical representations mentioned above directly displays uncertainty about attachment locations. More fundamentally, the ability to represent incompatible splits does not rescue these methods from their inherent sensitivity to taxon sampling that they obtain from considering only full splits[1]. Approaches that focus on partial splits are based on combining together various full splits that individually have low support into one partial split that has high support; the ability to display incompatible splits is an important improvement but is not a substitute for displaying partial splits. When clades have uncertain attachment points, consensus methods that display a small number of highly supported partial splits may produce graphs that are more easily interpretable than graphs that represent on a large number of full splits that may have low support (see figures 5f and 5g). Nevertheless, these two approaches are complementary rather than being alternatives, and are most appropriate to different types of uncertainty. Therefore, an ideal method would combine both approaches.

2 Representing topological uncertainty with multiconnected trees

In this paper, we introduce multiconnected tree graphs as a method for representing wandering taxa in phylogeny distributions (Figure 3). We now provide an informal definition of multiconnected graphs and multiconnected trees, followed by a description of how multiconnected trees represent topological uncertainty. A more mathematically rigorous definition then follows.

2.1 Multifurcating trees represent a collection of full splits

Leaf labelled multifurcating trees present a collection of full splits corresponding to their leaf taxa $\mathcal{L}$. The majority consensus tree summarizes a probability distribution on bifurcating trees in terms of a single multifurcating tree; many other consensus methods also rely on multifurcating trees in order to summarize uncertainty. We therefore summarize how multifurcations can be used to represent uncertainty. The extended majority consensus tree goes beyond the majority consensus tree by moving beyond the concept of multifurcating trees. However, the extended majority consensus also shares a number of characteristics with the majority consensus, such as representing trees as a collection of splits.

2.1.1 Bifurcating trees and multifurcating trees

Evolutionary trees are leaf-labelled bifurcating trees. We assume that evolutionary tree topologies are unrooted. No generality is lost by using unrooted trees because a root can be considered a

[1]Therefore pruning wandering taxa from consensus networks may substantially decrease the complexity of such a network, revealing a previously hidden tree structure. [*Make figure of 25S RNA example: Campylobacter has phylogenetic uncertainty: using a consensus network does not really show the underlying tree structure.*]

special leaf taxon in the unrooted framework. This allows us to consider a wandering root using the same framework we develop for wandering taxa.

The number of branches attached to a node is referred to as the *degree* of the node. Leaf nodes have degree 1 by definition. Internal nodes of degree d represent the division of a parent population into $d-1$ sub-populations. Because internal nodes of degree 2 do not partition the parent population into multiple sub-populations, they are superfluous. Internal nodes of degree 2 can be removed from unrooted topologies, and we only consider topologies on which they do not occur. A node of degree higher than 3 is called a *polytomy.* Topologies without polytomies are called *bifurcating* topologies because each internal node represents the bifurcation of a parent population into two sub-populations. Topologies with polytomies are termed multifurcating topologies.

We require that evolutionary tree topologies be bifurcating topologies and we note that a bifurcating tree with n leaf nodes always has $n-2$ internal nodes, n leaf branches, and $n-3$ internal branches. In contrast, a multifurcating topology has fewer internal nodes and internal branches.

2.1.2 Multifurcating trees represent phylogenetic uncertainty

In order to consider phylogenetic uncertainty, we must consider sets of trees that contain more than one tree. We say that a tree τ *extends* a tree μ if it is possible to change τ into μ by removing edges and merging their endpoints. We define $\langle\mu\rangle$ as the set of all bifurcating trees that extend μ. In this context, a polytomy in μ indicates uncertainty because there are a number of different bifurcating trees that extend it (See Figure 7). This kind of polytomy is thus called a *soft polytomy* to contrast it with the division of an ancestral population into multiple descendant populations (a *hard polytomy*). We will henceforth assume that all polytomies are soft.

The multifurcating tree that contains no internal branches is called the *star tree.* If μ is the star tree, then $\langle\mu\rangle$ is the entire space of bifurcating trees with n leaves and represents the absence of any information about the tree posterior.

Multifurcating trees are by nature well suited to represent certain kinds of uncertainties. However, uncertainty from wandering taxa or from short branches is not well represented by multifurcating trees. Therefore, instead of seeking an improved method of constructing multifurcating trees, we seek to extend the idea of the multifurcating trees to represent partial splits.

2.1.3 Internal branches and splits

The majority consensus tree makes use of the fact that multifurcating trees and bifurcating trees can be equivalently represented either as a graph or as a set of full splits that are induced by cutting each branch of the graph.

Removing a branch from a bifurcating or multifurcating tree divides the leaf taxa $\mathcal{L}$ into two non-empty subsets and so each branch of the tree is associated with a bi-partition $A = \{A_1, A_2\}$ of leaf taxa. In the context of phylogenetics, such a bi-partition is called a split, and is written $A_1|A_2$. We therefore define a split on leaf taxa $\mathcal{L}$ as a pair of non-empty sets A_1 and A_2 that are disjoint ($A_1 \cap A_2 = \emptyset$). If the split contains all the leaf taxa ($\mathcal{L} = A_1 \cup A_2$) then we call it a full split; otherwise we call it a partial split. If the split may or may not be a full split, then we just refer to it as a split.

If one half of a split contains only one taxon, then the split will be present in all trees. We therefore term such splits "uninformative" since they cannot be used to distinguish between different trees. If both halves of a split contain two or more leaf taxa, then the split can only be induced by an internal branch of the tree. Such splits are true of only some trees, and we term them "informative" splits.

Because each branch of a topology corresponds to a full split, we may therefore describe a tree μ in terms of the set $S(\mu)$ of full splits that it implies. Thus, each multifurcating tree μ corresponds to both a set of full splits $S(\mu)$ and a set of bifurcating trees $\langle\mu\rangle$ that extend μ:

$$S(\mu) \quad \leftarrow \mu \rightarrow \quad \langle\mu\rangle \,.$$

In order to define the relationship between the full splits $S(\mu)$ and the set $\langle\mu\rangle$, we associate each full split π with the set $\langle\pi\rangle$ of all bifurcating trees displaying π. For a set Σ of splits, we define $\langle\Sigma\rangle$ as

$$\langle\Sigma\rangle \quad \equiv \quad \bigcap_{\pi\in C} \langle\pi\rangle \,. \tag{1}$$

If $\langle\Sigma\rangle$ is empty, then this means that no bifurcating tree can display all the splits in Σ; in this case we say that Σ is not compatible. We note that bifurcating trees that display all splits of μ also extend μ:

$$\langle S(\mu)\rangle \quad = \quad \langle\mu\rangle \,. \tag{2}$$

If τ is a bifurcating tree, then it is the only tree that displays all of its bi-partitions, just as it is the only tree that extends itself:

$$\langle S(\tau)\rangle \quad = \quad \langle\tau\rangle = \{\tau\}. \tag{3}$$

As a result, we can decompose statements about a bifurcating tree τ into statements about its splits $S(\tau)$. We also note that any compatible set Σ of full splits corresponds to a multifurcating tree that displays these splits and no others, and we term this tree $M(\Sigma)$, so that $\mu = M(S(\mu))$.

Finally, note that we might consider Σ to represent either the tree $M(\Sigma)$ or the set $\langle\Sigma\rangle$, but we do not in fact have to make such a choice. This is because $M(\Sigma)$ represents not only Σ but also $\langle\Sigma\rangle$, since $M(\Sigma)$ is the only tree that is extended by every tree in $\langle\Sigma\rangle$.

$$\langle M(\Sigma)\rangle \quad = \quad \langle\Sigma\rangle \,.$$

2.1.4 Interpreting the majority consensus tree

For future convenience, we define Π_l as the set containing all splits with probability greater than l. Therefore, if $l \geq 0.5$ then

$$M_l \quad = \quad M(\Pi_l).$$

We note that M_l trees do not represent a level l credible **(fixme: Bayesian)** interval. While partitions in Π_l have probability greater than l when considered singly, their joint probabilities may be less than l. Since the M_l tree is the intersection of several such events its probability may be much smaller than l. We also note that when the the M_l tree is fully resolved (e.g. not bifurcating) it may not be the most probable tree.

Instead of representing a credible **(fixme: Bayesian)** interval, the $M_{0.5}$ tree can be better understood as representing the center of a tree distribution. This is because $M_{0.5}$ tree is the median tree, in the sense that it minimizes the expected Robinson-Foulds (RF) distance to a random tree τ (**?**). Additionally, we note that a random tree can be expected to agree with at least a fraction l of the internal branches of an M_l tree. This tree therefore minimizes a loss function and can be an optimal estimate in terms of Bayesian decision theory (Holder et al. 2008).

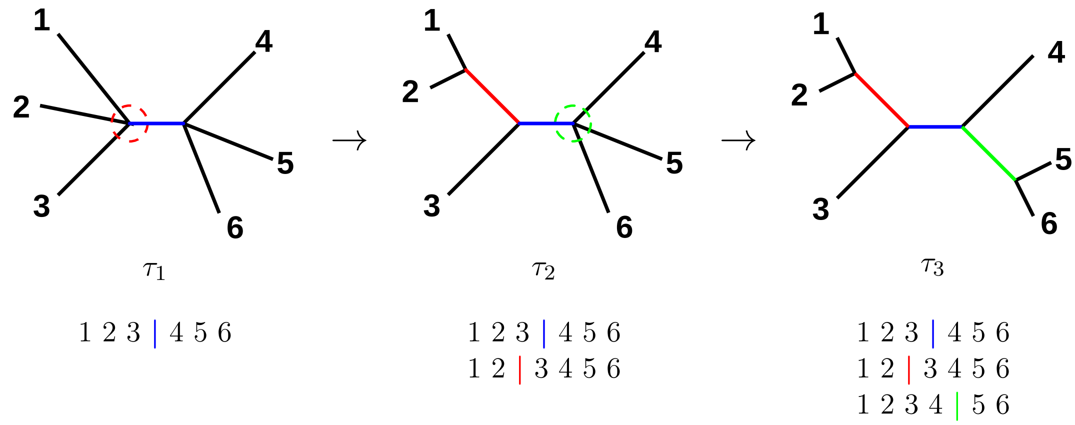


Figure 7: Extension. Adding more branches to a tree also increases the number of splits that the tree implies. The trees with the new branches contain no splits that are incompatible with the original tree, and are said to extend it, so that $\tau_1 \lhd \tau_2 \lhd \tau_3$. Polytomies are "soft" polytomies because they represent uncertainty about which of the alternative extensions is true, instead of representing speciation into three or more species.

2.2 Multiconnected Tree Sets

2.2.1 Informal Definition

Multiconnected trees represent a collection of splits. Just as in normal trees, each branch induces a split when cut. However, when a branch in a multiconnected tree is cut, it splits only leaf taxa that are on the same side of the branch in all embedded trees. Leaf taxa that may attach on alternate sides of the branch in different embedded trees are excluded from the split and are not partitioned by it. This produces a partial split. For example, in Figure 2d, the taxon X may attach on either side of the central branch (labelled in green), and occurs on different sides of this branch in the two embedded trees in Figures 2e and 2f. Therefore the split induced by the central branch in 2d is the partial split 12|34 which does not include the leaf taxon X.

Leaf taxa may wander individually, or may wander jointly in subtrees, as the clade $\{g, h\}$ in Figure 3 does. It is also possible for two different subtrees to wander across the same branch.

2.2.2 Precise Definition

Multiconnected graphs We first define a multiconnected graph as follows. A multiconnected graph $\mathcal{G} = (V, E, Q)$ consists of a set V of vertices, a multiset E of edges, and a set Q of equations. Each equation $q = (v, S)$ in Q consists of a vertex v and set S of vertices in V to which the vertex v may attach. Any multiconnected graph that may be obtained by equating v with some $s \in S$ and removing q from Q is said to be *embedded* in $\mathcal{G}$. We also consider any graph (V, E) to be identical to the multiconnected tree graph $(V, E, \emptyset)$.

We graphically depict an equation (v, S) in one of two ways. First, we may draw a dashed line from v to each endpoint $w \in S$ with an arrowhead pointing to w (Figure 5e). Second, we may draw cloud containing all and only the vertices S, and draw an arrowhead from v to the cloud (Figure 5f). Both types of figure represent the same underlying equation.

We associate each directed edge (u, v) in a multiconnected graph G a directed split $A_{u,v,G}$. We define $B_{u,v,G}$ as the set of leaf labels that are connected to vertex u but not vertex v when the edge

(u, v) is cut. We may then define the directed split $A_{u,v,G} \equiv B_{v,u,G}|B_{u,v,G}$. It is clear that when the graph G is a leaf-labelled tree with labels $\mathcal{L}$, the splits $\widehat{A_{u,v,G}}$ are in fact the full splits that would be obtained by cutting the branch (u, v).

Multiconnected trees We define a multiconnected tree as a multiconnected graph in which every embedded graph is a tree. Recall that a tree is defined as a graph with a single connected component that contains no cycles. In this paper we focus on the set of splits associating with a multiconnected tree, because we seek to use multiconnected trees to represent a set of supported splits, possibly including partial splits. We do not, however, characterize here the requirements for a multiconnected graph to be a multiconnected tree. Instead, we show that multiconnected graphs can be generated from certain collections of splits and that such multiconnected graphs are indeed multiconnected trees.

Proving the correspondence between split sets and their graphs Before we can present multiconnected trees and their graphs as useful tools for summarizing phylogenetic uncertainty, we must show that such graphs actually represent a collection of trees, and that split sets that satisfy the criteria for a multiconnected tree can be represented by such a graph. Both of these problems are comparatively trival for tree graphs and sets of full splits, making the majority consensus tree a useful summary of phylogenetic uncertainty. However,

3 Unrepresentable split collections

While any combination of full splits that is pairwise compatible may be displayed in a tree, only a certain partial split collections may be displayed in this way.

4 Multiconnected Split Sets

4.1 Representations of a multiconnected split sets

The majority consensus tree can be represented in three equivalent ways. It can be represented as a collection Σ of splits that are pairwise compatible, or as a collection $\langle\Sigma\rangle$ of trees that display all splits in Σ, or as a graph $M(\Sigma)$ that every tree in $\langle\Sigma\rangle$ extends (See figure 8). Also, each edge of the graph $M(\Sigma)$ corresponds to a split contained in Σ.

Likewise, an extended majority consensus tree be represented in three analogous ways. It can be represented as a collection of partial splits Σ that satisfy extended compatibility rules, or as a collection of trees $\langle\Sigma\rangle$ that display all splits in Σ, or as a graph $G(\Sigma)$ that every tree in $\langle\Sigma\rangle$ extends/refines. The graph $G(\Sigma)$ has two types of edges; type 1 edges (thick lines) correspond to splits or partial splits. Type 2 edges (dashed lines) connect an endpoint of a type 1 edge to its possible attachment points (See figure 9). The graph $G(\Sigma)$ also embeds all bifurcating trees in $\langle\Sigma\rangle$, and therefore represents common information in $\langle\Sigma\rangle$, instead of representing Σ directly. Therefore, in order for $G(\Sigma)$ to represent Σ, $\langle\Sigma\rangle$ must also represent Σ, and there must not be any other set Σ' such that $\langle\Sigma\rangle = \langle\Sigma'\rangle$.

4.2 Definition of a multiconnected split sets

We define a multiconnected tree as set of partial splits Σ that are pairwise compatible and satisfy additional constraints. These constraints prevent the partial splits in Σ from interacting to jointly

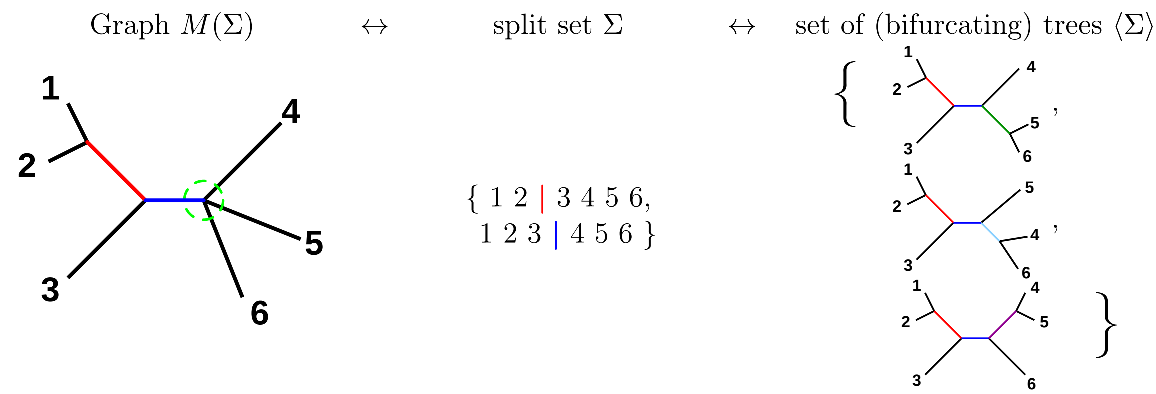


Figure 8: Three representations of multifurcating trees all contain the same information. The first representation is a graph. The second representation is a set of compatible splits. The third representation is the set of fully resolved trees that contain all splits in the split set. They also extend the graph.

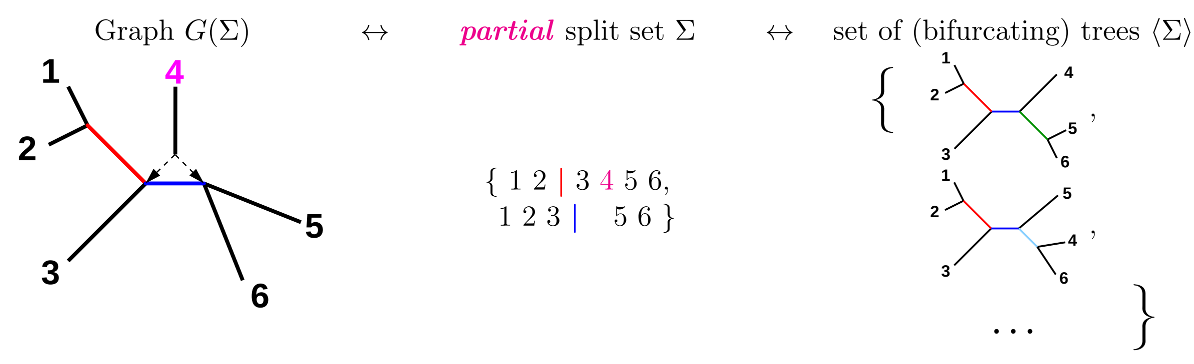


Figure 9: Multiconnected trees can also be represented in three ways that contain the same information. The graph now has multiple attachment points. The split set now contains partial splits. The set of fully resolved trees is still a set of fully resolved trees. (**Emphasize "..." ?** as "3(?) more")

imply any new quartets that are not part of splits in Σ. These constraints reduce to the requirement of standard pairwise compatibility if all splits in Σ are in fact full splits. To describe these constraints, we begin by introducing the concept of ordered splits and by defining three new relations between ordered splits.

4.2.1 Pairwise Compatibility

For any two (possibly partial) splits $A = A_1|A_2$ and $B = B_1|B_2$ we define $\#(A,B)$ as the number of sets $A_i \cap B_j$ that are non-empty. This number can range from 0 to 4. If $\#(A,B) = 4$ then $\langle\{A,B\}\rangle = \emptyset$ and $\{A,B\}$ is incompatible; otherwise A and B are compatible. If A and B are distinct full splits that are compatible, then $\#(A,B)$ will always be 3. However, if A and B are different partial splits, then $\#(A,B)$ may be smaller than 3.

4.2.2 Ordered splits

If undirected[2] branches on a tree correspond to the splits that they induce, it is natural to seek a correspondence between directed branches and splits. We do this by introducing the concept of ordered splits. When a split A is an ordered split, then we distinguish between $A_1|A_2$ and $A_2|A_1$. We say that $A_1|A_2$ points towards the part of the tree containing the leaf nodes A_2. In order to refer to a split that is the same as a split A but points in the opposite direction, we will use the notation $A^t = A_2|A_1$. (**small figure? Jeff: Yes**)

4.2.3 The "implies" relation

We say that $A \implies B$ iff ($B_1 \subseteq A_1$ and $B_2 \subseteq A_2$) or ($B_1 \subseteq A_2$ and $B_2 \subseteq A_1$). For example: 123|456 implies 12|56.

4.2.4 The "left of" relation

We define the relation $A < B$ on any two ordered splits A and B to mean $A_1 \subset B_1$ and $B_2 \subset A_2$ (see Figure 10). The "<" relation is sensitive to order of both A and B; however, we note that $A < B$ is true if and only if $B^t < A^t$, so that these two expressions are equivalent. If $A < B$ then we say that A is "to the left of" B. The geometric interpretation of the $<$ relation is that A and B are both pointing in the same direction, and that A points to B but B does not point to A. As a example of the "<" relation, consider the splits 12|5Y34 $<$ 12X5|34. These splits satisfy the $<$ relation because $12 \subset 12X5$ and $34 \subset 5Y34$ (See figure 10).

We also note some algebraic properties of the $<$ relation. First, we cannot have $A < A$ because the subsets are strict. Second, if $A < B$, then it is not the case that $B < A$ because this would require that A_1 and B_1 are strict subsets of each other. Third, it is transitive: $A < B$ and $B < C$ together imply that $A < C$. These three properties mean that "<" is a strict partial order on splits.

We note that if $A < B$ then $\#(A,B)$ must be at least 2 because of the two subset relationships. Further, $A_1 \cap B_2$ must be empty, because $A_1 \cap A_2$ must be empty, and $B_2 \subset A_2$. Therefore, $\#(A,B)$ cannot be 4, and A and B must be compatible. We distinguish two cases. If $A_2 \cap B_1$ is empty, then $\#(A,B)$ equals 2, and we may write $A <_2 B$. This cannot occur for full splits[3]. If $A_2 \cap B_1$ is not empty, then $\#(A,B)$ equals 3, and we may write $A <_3 B$. We note that the $<_2$ relation imposes a weaker ordering than the $<_3$ relation. Specifically, if two full splits α and β imply A and B respectively, then $A <_3 B$ implies that $\alpha < \beta$. However, $A <_2 B$ implies either $\alpha < \beta$ or $\beta < \alpha$.

[2] Jeff: defined?

[3] Why? Show.

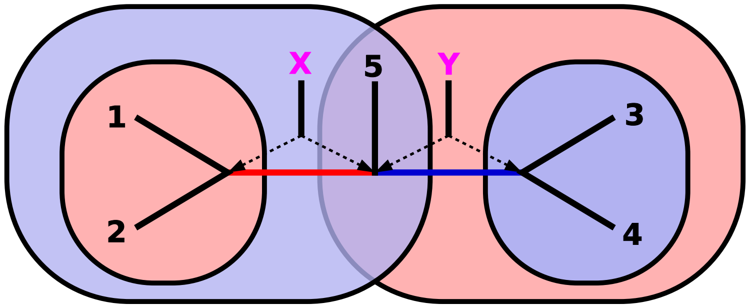


Figure 10: Relation: Left of. The multiconnected tree graph here illustrates $12|5Y34 < 12X5|34$ as an example of the " $<$ " relation. This relation satisfies the definition of " $<$ " because $12 \subset 12X5$ and $34 \subset 5Y34$. (i) The partial split $12|5Y34$ is illustrated by the two red-shaded ellipses containing 12 and $5Y34$ respectively. The split $XY|1234$ also corresponds to the red branch of the multiconnected tree graph. (ii) The partial split $12X5|34$ is illustrated by the two blue-shaded ellipses containing $12X5$ and 34 respectively. The split $12X5|34$ also corresponds to the blue branch of the multiconnected tree graph.

We additionally define $A \ll B$ in Σ if (i) $A < B$ and (ii) there is no $C \in \Sigma$ such that $A < C$ and $C < B$ and (iii) A and B^t do not directly wander[4] over any other split in Σ.

4.2.5 The "wanders over" relation

We define the relation $A \downarrow B$ on any two ordered splits A and B such that $A \downarrow B$ if and only if $B_1 \cup B_2 \subseteq A_2$ (See Figure 11). In this case we say that A wanders over B. This relation is sensitive to the ordering of A, but not to the ordering of B. This is because the two endpoints of a branch may wander over different other branches, so that A and A^t need not wander over the same splits. This relation allows a single taxon to wander over a branch, in which case A represents a leaf branch, as in the example $X|12345Y \downarrow 12|345$. However, clades may also wander, as in the case: $XY|12345 \downarrow 12|345$. We also write $A \not\downarrow B$ to indicate that A does not wander over B.

We also note a few important algebraic relationships with geometric interpretations. First, we cannot have that $A \downarrow A$, because A_1 and A_2 must be disjoint and so A_1 cannot be a subset of A_2. Second, if $A \downarrow B$ then it cannot be the case that $B \downarrow A$. Therefore two branches cannot wander over each other. Third, if $A \downarrow B$ and $B \downarrow C$, then $A \downarrow C$; the wandering relationship is therefore transitive. These three properties mean that $\downarrow$ is a strict partial order on splits. Fourth, if $A \downarrow B$, then it cannot be the case that $A^t \downarrow B$; the two different ends of a branch cannot wander over the same branch. Finally, if $A < B$ and $B \downarrow C$, then $A \downarrow C$; if B wanders over C then any branches that are in the subtree behind[5] B also wander over A.

This last point indicates that a split A that wanders over a split C may in fact be separated from C by other splits B in the collection Σ of splits. We therefore define a new relation $A \Downarrow C$ to indicate that $A \downarrow C$ and there is no split $B \in \Sigma$ such that $A < B$ and $B \downarrow C$. In this case we say that A wanders *directly* over C in Σ. This fact is important in the construction of the multiconnected tree graph $G(\Sigma)$ from the split set Σ.

[4] Wandering (and direct wandering) not defined yet. Move definition of $\ll$ and $\Downarrow$ to a section on relations that are relevant to graph construction?

[5] Jeff: what does this mean? Perhaps just mention that this relationship will become important when we begin to construct the graph.

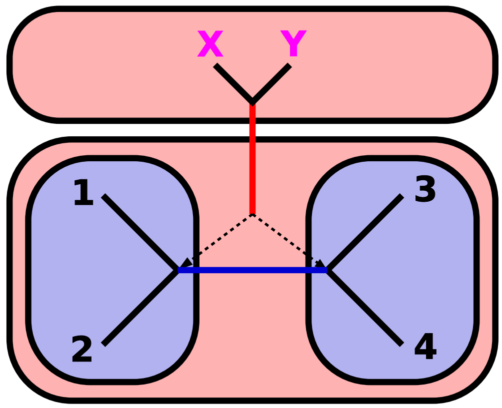


Figure 11: Relation: Wandering. The multiconnected tree graph here illustrates $XY|1234 \downarrow 12|34$ as an example of the "↓" relation. (i) The full split $XY|1234$ is illustrated by the two red-shaded ellipses containing XY and 1234 respectively. The split $XY|1234$ also corresponds to the red branch of the multiconnected tree graph. The red branch has two alternate attachment points, indicated by dotted lines, which is why it is called a wandering branch. (ii) The partial split 12|34 is illustrated by the two blue-shaded ellipses containing 12 and 34 respectively. The partial split 12|34 also corresponds to the blue branch of the multiconnected tree. Because 12|34 does not contain X or Y, the clade XY may attach at either side of the blue branch. (iii) We also note that the directed split X|1234Y wanders over the split 12|34. However, it does not wander *directly* over 12|34 in this illustration because XY|1234 is in between in the sense that X|1234Y<XY|1234 and XY|1234 ↓ 12|34. **(Jeff wonders if you can't just write XY↓12|34 instead of XY|1234↓12|34. Well, this is supposed to be a compatibility relation that corresponds to the graph..., and this change would remove the compatibility nature...)**

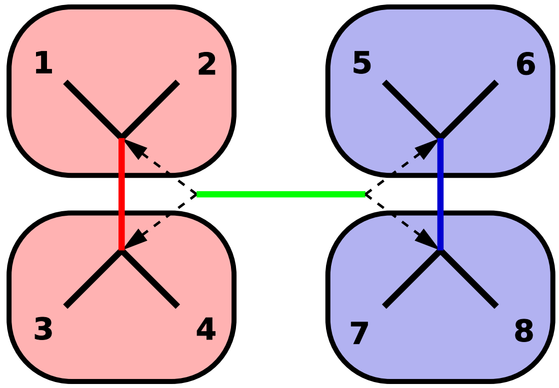


Figure 12: Relation: Disjoint. We write $A \perp B$ when the leaf sets of A and B do not overlap, but there is a third branch C such that $C_1|C_2 \downarrow A$ and $C_2|C_1 \downarrow B$. In this example A =12|34, B=56|78, and C=1234|5678.

4.3 The relation "disjoint": "$\perp$"

We define the relation $A \perp B$ on any two splits A and B such that $A \perp B$ if and only if $(A_1 \cup A_2) \cap (B_1 \cup B_2) = \emptyset$. (See figure 12).

4.4 Rules

A multiconnected tree on leaf set $\mathcal{L}$ is a collection of unordered splits Σ such that

- Σ contains the leaf branches $x|\mathcal{L} - x$ for every $x \in \mathcal{L}$.
- If $A \in \Sigma$ then $A_1 \cup A_2 \subseteq \mathcal{L}$.
- If $A, B \in \Sigma$ then A and B must relate through "$<$", "$\downarrow$", or "$\perp$".

Rule 1: Related through "$<$" Any two ordered splits A and B may be related through "$<$", in which case one of the following must be true:

1. $A < B$ (same as $B^t < A^t$)
2. $B < A$ (same as $A^t < B^t$)
3. $A^t < B$ (same as $B^t < A$)
4. $B < A^t$ (same as $A < B^t$)

Rule 2: Related through "$\downarrow$" Any two ordered splits A and B may be related through "$\downarrow$", in which case one of the following must be true:

1. $A \downarrow B$
2. $A^t \downarrow B$
3. $B \downarrow A$
4. $B^t \downarrow A$

Rule 3: Related through "$\perp$" Any two ordered splits A and B may be related through "$\perp$". Because the relation "$\perp$" is not sensitive to the ordering of A and B it is not necessary to consider cases for this relation.

Additional facts Firstly, we claim that each of these nine cases is exclusive.

Finally, we note that if all splits in Σ are full splits (that is, when Σ represents a multifurcating tree) then all pairs of splits satisfy Rule 1.

4.4.1 The meaning of the rules

The rules above lead to a simple geometric interpretation.

1. The first rule prohibits the insertion of branches that represent partial splits missing taxa $X_1, X_2, \ldots$ in a region of the tree that is separated from $X_1, X_2, \ldots$. Equivalently, the branches over which a taxon x wanders must not be separated from each other, or from the attachment point of x by any branches over which x does *not* wander[6].

2. The second rule ensures that each wandering branch can independently be attached to any node of the multiconnected tree graph, without restricting the attachment points of other wandering branches. ???

3. The third rule ensures allow both ends of a branch to wander, and ensures that the tree is connected in this case.

4.5 Examples and counter-examples

Example 1. 12|34 and 12|56

This example is prohibited. Each branch wanders over the other, and therefore their wandering is not independent. The constraint that wandering branches must relate through Rule 2 prohibits this kind of pair, but gains the property that wandering branches can decide where to attach without affecting the attachment points of other wandering branches.

Example 2. A=12X|3456 and B=1234|56

This example is prohibited. When considering the multiconnected tree with just split A as an internal split, we find that unplugging X resulting in the addition of a new branch splitting a node that is *different* that the node from which X was removed. As a result, this combination is note closed, but jointly implies the full split 1234X|56. Thus, an attempt to display these splits in a graph, actually results in displaying a different collection of splits: A=12X|3456 and B'=1234X|56.

5 Proofs

We claim that if Σ is a collection of partial splits that obeys the rules, then:

1. Σ is compatible. That is, $\langle\Sigma\rangle$ is not empty.

2. $\langle\Sigma\rangle$ is fully closed. That is $q\,\langle\Sigma\rangle = \bigcup_{\sigma\in\Sigma} q(\sigma) = q(\Sigma)$.

[6] Well, given that we have 3 overlaps for the splits A and B, this rule prevents a taxon x that is not in B from being separated from B by A, which would

3. It is always possible to construct a graph (the multiconnected tree graph) that represents exactly the splits in Σ.

These properties are both necessary for the multiconnected tree graph to exist. If the first property is not satisfied, then no tree exists that is compatible with Σ, and so the graph cannot exist.

We note that the graph represents Σ by embedding a set of trees that have exactly the splits Σ in common. The second property is required for such a set of trees to exist; if it is not satisfied, then any set of trees that is compatible with Σ will also be compatible with other splits not in Σ, and so there will be no set of of trees that represents the splits Σ.

5.1 Proof that a multiconnected tree is compatible

This proof proceeds by noting that one can take a pair of branches $A \downarrow B$ and extend B to $B' = B_1|A_1 + B_2$ (Theorem 111). A no longer wanders over this new split B, but instead $A < B'^t$. Now, the new collection Σ' that is obtained by replacing B by B' is also a multiconnected tree (Theorem 100). Clearly $\langle\Sigma'\rangle \subset \langle\Sigma\rangle$ so if we can show that Σ' is compatible, then Σ is compatible as well. This procedure also increases the support $B_1 \cup B_2$ of one of the partial splits, and so successive applications of this procedure eventually yield a multiconnected tree composed only of full splits. Such a tree must be a multifurcating tree, and therefore is compatible. Therefore the original

5.2 Proof that a multiconnected tree is fully closed

Consider any quartet x that is not implied by any of the individual splits $\sigma \in \Sigma$. If we can always resolve Σ to a multifurcating tree that does not imply x, then $x \notin q\,\langle\Sigma\rangle$ and Σ does not jointly imply any quartets that are not part of its splits.

We can do this by considering each branch $\sigma \in \Sigma$ and showing that σ can be extended to form σ' that does not imply x. However this proof is messy and requires a lot of cases. Surely there is an easier way?

6 The multiconnected tree graph

6.1 How to construct the graph $G(\Sigma)$ from Σ

6.1.1 The graph with ranges

The multiconnected tree graph $G(\Sigma)$ can be constructed from any multiconnected split set Σ as follows.

1. For each ordered split A in $\check{\Sigma}$, add a new vertex $V(A)$ to $G(\Sigma)$ that is associated with A. Now if Σ contains some positive number $|\Sigma|$ of splits, then $G(\Sigma)$ contains $2|\Sigma|$ vertices.

2. For each split A in Σ create an (undirected) edge that connects the vertices $V(A)$ with $V(A^t)$. This edge corresponds to the (undirected) split A, and its two endpoints correspond to different orientations of A.

3. For each split $A \in \Sigma$, we label the vertex $V(A)$ a non-wandering vertex if there is no $B \in \Sigma$ such that $A \Downarrow B$.

4. For each pair of non-wandering vertices $V(A)$ and $V(B)$, if $A < B^t$ and there is no split C such that $A < C < B^t$, we identify $V(A)$ with $V(B)$. (Note that elsewhere we call this situation "$A \ll B$ in Σ") (See Lemma 83 for the transitivity of this identification criterion.)

5. For each wandering vertex $V(A)$ we construct an equation (v, S) where $v = V(A)$. We add to S the vertex $v(B)$ for any edge B such that $A \Downarrow B$. Every vertex in S indicates an alternative attachment point for $V(A)$. When attempting to graphically depict an equation, we may draw a dashed line from v to every node $s \in S$, or we may draw a colored cloud encircling the points S and connect v to the perimeter of the cloud.

If Σ contains only full splits, then there are only non-wandering vertices. In such a case, $G(\Sigma)$ will create a tree, where each edge is associated with the split that it induces.

6.1.2 The graph with multi-edges

In order to construct the graph $G(\Sigma)$ from Σ we first define the nodes of graph $G(\Sigma)$

Definition 3. For any set of ordered splits Σ we define the set $NW(\Sigma) \equiv \{A \in \Sigma : A \not\Downarrow B \text{ for all } B \in \check{\Sigma}\}$ of ordered splits that do not wander in Σ.

Definition 4. We define the relation $N(\Sigma)$ as $\{(A, B) \in \check{\Sigma}^2 : A = B \vee A \lll B^t\}$. Note that $A \lll B^t$ requires that $A, B \in NW(\hat{\Sigma})$. The fact that $(A, B) \in N(\Sigma)$ will indicate that the ordered splits A and B point to the same node in $G(\Sigma)$, the multiconnected graph of Σ.

Lemma 5. *If Σ is a multiconnected split set, then the relation $N(\Sigma)$ forms an equivalence relation.*

Proof. First, the relation clearly satisfies $A \sim A$.

Second, suppose that $A \sim B$. If $A = B$ then clearly $B \sim A$. Alternatively, if $A \neq B$, then $A \lll B^t$, and so $A < B^t$. Also, there is no $C \in \Sigma$ such that $A < C < B^t$. Transposing everything, this then implies that $B < A^t$ and there is no $C \in \check{\Sigma}$ such that $B < C^t < A^t$. This then implies that $B \lll A^t$, and thus $B \sim A$.

Finally, suppose that $A \sim B$ and $B \sim C$.

- If either $A = B$ or $B = C$, then either $A \sim B$ or $B \sim C$ implies $A \sim C$.
- If $A \neq B$ and $B \neq C$, then $A \lll B^t$ and $B \lll C^t$. But then $B \lll A^t$ and $B \lll C^t$, and so $A \lll C^t$ by Lemma 83.

Therefore $A \sim B$ and $B \sim C$ together imply $A \sim C$. □

Definition 6. We define the node pointed to by the split $A \in NW(\check{\Sigma})$ as the equivalence class of A under $N(\Sigma)$:

$$\nu(A, \Sigma) \quad = \quad \left\{B \in \check{\Sigma} : (A, B) \in N(\Sigma)\right\}.$$

Definition 7. If Σ is a multiconnected split set, then we define $V(\Sigma)$ as the set of equivalence classes of $N(\Sigma)$. Thus,

$$V(\Sigma) \quad \equiv \quad \bigcup_{A \in NW(\check{\Sigma})} \nu(A).$$

The set $V(\Sigma)$ will be the vertices of the graph $G(\Sigma)$.

Definition 8. If Σ is a multiconnected split set and $A \in \check{\Sigma}$ then we define the set $T(A, \Sigma)$ of targets of A as

$$T(A, \Sigma) \quad = \quad \begin{cases} \{\nu(A)\} & \text{if } A \in NW(\check{\Sigma}) \\ \bigcup_{B \in NW(\check{\Sigma}) : A \Downarrow B} \nu(B) & \text{if } A \notin NW(\check{\Sigma}) \end{cases}$$

Lemma 9. $|T(A,\Sigma)| > 1$ *if and only if* $A \notin NW(\check{\Sigma})$.

Proof. $|T(A,\Sigma)| > 1$ contradicts $A \in NW(\check{\Sigma})$, since $A \in NW(\check{\Sigma})$ requires $T(A,\Sigma) = \{\nu(A)\}$ and thus requires $|T(A,\Sigma)| = 1$. Therefore, $|T(A,\Sigma)| > 1$ implies $A \notin NW(\check{\Sigma})$.

Now, if $A \notin NW(\check{\Sigma})$, then the set of splits $\{C \in \check{\Sigma} : A \Downarrow C\}$ cannot be empty. Let us consider the order $A < B \equiv A \downarrow B \vee A^t \downarrow B$. This is a partial order because $A \not< A$, $A < B$ and $B < C$ implies $A < C$, and $A < B$ implies $B \not< A$. There must be a maximal element D in $\{C \in \check{\Sigma} : A \Downarrow C\}$ under this order. But then D and D^t are both in $NW(\Sigma)$. Therefore $\nu(D,\Sigma)$ and $\nu(D^t,\Sigma)$ are both in $T(A,\Sigma)$. Since these must be distinct vertices, and both must be in $T(A,\Sigma)$, we have that $|T(A,\Sigma)| > 1$. □

Definition 10. If Σ is a multiconnected split set and $A \in \check{\Sigma}$ then the directed edge associated with A is $e(A,\check{\Sigma}) \equiv (T(A^t,\Sigma), T(A,\Sigma))$. If $A \in \Sigma$ then $e(A,\Sigma)$ is the undirected version of this.

Definition 11. If Σ is a multiconnected split set, then we define the multiset $E(\Sigma)$ as

$$E(\Sigma) \quad = \quad \bigcup_{A \in \Sigma} e(A).$$

Lemma. *The multiset* $E(\Sigma)$ *contains each edge only once. That is, if* A *and* B *are distinct members of* $\check{\Sigma}$*, then* $e(A) \neq e(B)$.

Proof. Suppose $e(A) = e(B)$. Then $T(A,\Sigma) = T(B,\Sigma)$ and $T(A^t,\Sigma) = T(B^t,\Sigma)$. Note that by Lemma 9, $T(\cdot,\Sigma)$ can be equal for two distinct splits if and only if they are both wandering, or both non-wandering.

If A and A^t are in $NW(\Sigma)$, then this requires $\nu(A,\Sigma) = \nu(B,\Sigma)$, and $\nu(A^t,\Sigma) = \nu(B^t,\Sigma)$. But then $A < B^t$ and $A^t < B$. These cannot both be true, so we cannot have $e(A) = e(B)$ if $\{A, A^t\} \subseteq NW(\Sigma)$.

If $A^t \in NW(\Sigma)$ and $A \notin NW(\Sigma)$, then we have $A^t \lll B$ and there exists some element C such that $A \Downarrow C$ and $B \Downarrow C$. But then $A^t < B \downarrow C$ and so $A^t \downarrow C$. But then $C \in A_1$ and $C \in A_2$, which is a contradiction. Therefore we cannot have $e(A) = e(B)$ if $A^t \in NW(\Sigma)$ and $A \notin NW(\Sigma)$.

If $A \in NW(\Sigma)$ and $A^t \notin NW(\Sigma)$ then, we cannot have $e(A^t) = e(B^t)$ by the previous case.

If $A^t, A \notin NW(\Sigma)$ then we must have some $C, D \in \Sigma$ such that $A \Downarrow C$ and $B \Downarrow C$ and $A^t \Downarrow D$ and $B^t \Downarrow D$. But by Lemma 78, we must have either $A < B$ or $B < A$. If $A < B$ then $A < B \downarrow C$, which contradicts $A \Downarrow C$. If $B < A$, then $B < A \downarrow C$, which contradicts $B \Downarrow C$. Thus, we cannot have $e(A) = e(B)$ if $A^t, A \notin NW(\Sigma)$.

Therefore, for A and A^t either wandering or non-wandering, $e(A) \neq e(B)$. Therefore, if $A \neq B$ then $e(A) \neq e(B)$. □

Definition 12. If Σ is a multiconnected split set, then we define the multiconnected graph $G(\Sigma)$ associate with Σ as

$$G(\Sigma) \quad \equiv \quad (V(\Sigma), E(\Sigma)).$$

6.2 Facts about nodes

Lemma 13. *If* Σ *is a multiconnected split set and* $n \in V(\Sigma)$*, then there exists some* $A \in NW(\check{\Sigma})$ *such that* $n = \nu(A,\Sigma)$.

Proof. By the definition of $V(\Sigma)$, every element n is $\nu(A,\Sigma)$ for some $A \in NW(\check{\Sigma})$. □

Lemma 14. *If Σ is a multi-connected split set and $n \in V(\Sigma)$, then n is not empty.*

Proof. For any $n \in V(\Sigma)$, there is some $A \in NW(\check{\Sigma})$ such that $n = \nu(A, \Sigma)$. But then $A \in n$, and n is not empty. $\square$

Definition 15. Let the set $S(A, \Sigma) \equiv \left\{B \in \check{\Sigma} : A < B \text{ or } A \downarrow B\right\}$. That is, $S(A, \Sigma) \equiv \left\{B \in \check{\Sigma} : A <_o B\right\}$ (See 82 for a definition of $<_o$). Since $A \notin S(A, \Sigma)$, we note that $S(A, \Sigma) = S(A, \Sigma \cap \{A\}^C)$. Therefore, if $A \notin \Sigma$, then $S(A, \Sigma + A) = S(A, \Sigma)$.

Lemma 16. *If A does not directly wander in Σ then for any $M \in \min_{<_o} S(A, \Sigma)$, $A < M$.*

Proof. Suppose that $A \downarrow M$. But since A does not directly wander in Σ, there must be some element B of $\check{\Sigma}$ such that $A < B \downarrow M$. But then $B \in S$ and $B <_o M$, contradicting the fact that M is a minimal element of $S(A, \Sigma)$. Therefore, $A \not\downarrow M$ and so $A < M$. $\square$

Lemma 17. *If Σ is a multiconnected split set and A does not directly wander in Σ and $B \in S(A, \Sigma)$ but $B \notin \min_{<_o} S(A, \Sigma)$, then there is some $M \in \min_{<_o} S(A, \Sigma)$ such that $A < M <_o B$.*

Proof. By the definition of minimal elements, if $B \notin \min_{<_o} S(A, \Sigma)$, then there is some $M \in \min_{<_o} S(A, \Sigma)$ such that $M <_o B$. By Lemma 16, we must have $A < M$. $\square$

Theorem 18. *If Σ is a multiconnected split set and $A \in NW(\check{\Sigma})$ and $M^t \in min_{<_o} S(A, \Sigma)$ and $M \in NW(\Sigma)$, then $M \in \nu(A, \Sigma)$.*

Proof. If $M^t \in \min_{<_o} S(A, \Sigma)$ then $A < M^t$ by Lemma 16. There cannot be any element $B \in \check{\Sigma}$ such that $A < B < M^t$ because M^t is a $<_o$-minimal element of $S(A, \Sigma)$. If $M \in NW(\Sigma)$ then $A \ll M^t$ by the definition of $\ll$. Therefore $M \in \nu(A, \Sigma)$.

$\square$

Claim 19. If $B \in S(A, \Sigma)$ then either $B \in \min S(A, \Sigma)$ or there is some $M_1 \in \min S$ such that $M_1 <_o B$.

Lemma 20. *If $M_1, M_2 \in \min S(A, \Sigma)$ and $M_1 \neq M_2$ then either $M_1^t < M_2$ or $M_1^t \downarrow M_2$ or $M_2^t \downarrow M_1$.*

Proof. Since $A < M_1$ and $A < M_2$ by Lemma 16, then by Lemma 80 we must have either

- $M_1 < M_2$
- $M_2 < M_1$
- $M_1^t < M_2$
- $M_1^t \downarrow M_2$
- $M_2^t \downarrow M_1$

Now $M_1 < M_2$ and $M_2 < M_1$ cannot occur since M_1 and M_2 are minimal elements of $S(A, \Sigma)$ according to $<_o$. The remaining cases are exactly those claimed. $\square$

Theorem 21. *If $M \in \min_{<_o} S(A, \Sigma)$ and $M^t \notin NW(\check{\Sigma})$ then there exists some $C \in S(A, \Sigma)$ such that $M^t \Downarrow C$.*

Proof. Suppose that $M^t \Downarrow C$. Now since $M^t < A^t$, then by Lemma 76 either

- $C < A^t$

- $C^t < A^t$
- $A^t \downarrow C$

  In this case we have $M^t < A^t \downarrow C$, which contradicts $M^t \Downarrow C$.
- $A \downarrow C$
- $A \perp C$ and there is some D such that $D^t \downarrow A$ and $D \downarrow C$ and $M^t < D$.

  In this case we have $M^t < D \downarrow C$, which contradicts $M^t \Downarrow C$.

Therefore, any such C has either $A < C^t$ or $A < C$ or $A \downarrow C$. Thus, either C or C^t is in S. Since $M \Downarrow C$ and $M \Downarrow C^t$, the theorem is proved. □

Theorem 22. *If $M_1 \in \min_{<_o} S(A, \Sigma)$ and $M_1^t \notin NW(\check{\Sigma})$ then there exists some $M_2 \in \min_{<_o} S(A, \Sigma)$ such that $M_1^t \Downarrow M_2$.*

Proof. By Theorem 21 there is some element $B \in S(A, \Sigma)$ such that $M_1^t \Downarrow B$. Then there must be some $M_2 \in \min S$ such that either $M_2 = B$ or $M_2 <_o B$.

Case 1. Now, suppose that $M_2 < B$. Then by Lemma 75 either $M_1^t \Downarrow M_2$ or $M_2 < M_1$. The latter cannot occur since M_1 and M_2 are both in $\min_{<_o} S(A, \Sigma)$, so in this case $M_1^t \Downarrow M_2$.

Case 2. Suppose that $M_2 \downarrow B$. Then by Lemma 77 we must have one of

- $M_1^t \downarrow M_2$
- $M_2 \downarrow M_1^t$

  Disallowed by Lemma 20.
- $M_1^t < M_2$

  This implies $M_1^t < M_2 \downarrow B$, which would contradict $M_1^t \Downarrow B$.
- $M_2 < M_1^t$

  Disallowed by Lemma 20.
- $M_2 < M_1$

  Disallowed by Lemma 20.

Thus in this case, we must have $M_1^t \downarrow M_2$. Now suppose that there is some split $Z \in \check{\Sigma}$ such that $M_1^t < Z \downarrow M_2$. then $M_1^t < Z \downarrow B$ since $M_2 \downarrow B$. But this contradicts $M_1^t \Downarrow B$. Therefore there is no such split Z and $M_1^t \Downarrow M_2$.

Case 1. Finally, we suppose that $M_2 = B$.

In each case, we see that $M_1^t \Downarrow M_2$. □

Theorem 23. *If Σ is a multiconnected split set and $A \in NW(\check{\Sigma})$, then $\nu(A, \Sigma) = \{A\} \cup \max_{\downarrow} (\min_{<_o} S(A, \Sigma))^t$.*

Proof. For any element $N \in \max_{\downarrow} (\min_{<_o} S(A, \Sigma))^t$, N does not wander over any element of $(\min_{<_o} S(A, \Sigma))^t$ because N is maximal. Therefore, by Lemma 22, N does not directly wander in Σ. Since $N^t \in \min_{<_o} S(A, \Sigma)$ and $N \in NW(\Sigma)$, Lemma 18 implies $N \in \nu(A, \Sigma)$. Therefore,

$$\{A\} \cup \max_{\downarrow} \left(\min_{<_o} S(A, \Sigma) \right)^t \quad \subseteq \quad \nu(A, \Sigma).$$

Now, any element of $B \in \nu(A, \Sigma)$ is either equal to A, or has $A \ll B^t$. In either case $B \in NW(\check{\Sigma})$. Clearly if $B = A$, then $B \in \{A\} \cup \max_{\downarrow}(\min_{<_o} S(A,\Sigma))^t$.

Now suppose $B \neq A$ and so $A \ll B^t$. Then $B^t \in S(A,\Sigma)$ because $A < B^t$. Suppose $B^t \notin \min_{<_o} S(A,\Sigma)$. Then there must be some $M \in \min_{<_o} S(A,\Sigma)$ such that $A < M < B^t$ or $A < M \downarrow B^t$. $A < M < B^t$ cannot occur because $A \ll B^t$. $A < M \downarrow B^t$ cannot occur because this would imply $A \downarrow B^t$. Therefore, $B \in (\min_{<_o} S(A,\Sigma))^t$.

Now suppose that $B \notin \max_{\downarrow}(\min_{<_o} S(A,\Sigma))^t$. Then $B \downarrow C$ for some $C \in (\min_{<_o} S(A,\Sigma))^t$. Furthermore, since $B \in NW(\check{\Sigma})$, there must be some $D \in \check{\Sigma}$ such that $B < D \downarrow C$. Since $A < C$, Lemma 74 implies that either $A < D^t$ or $D \downarrow A$. Now, if $A < D^t$ then $A < D^t < B^t$, which contradicts $A \ll B^t$. Alternatively, if $D \downarrow A$, then $B < D \downarrow A$ and $B \downarrow A$, which contradicts $A \ll B^t$. Therefore, we must have $B \in \max_{\downarrow}(\min_{<_o} S(A,\Sigma))^t$ for any $B \in \nu(A,\Sigma)$ for which $A \neq B$. Therefore,

$$\nu(A,\Sigma) \quad \subseteq \quad \{A\} \cup \max_{\downarrow}\left(\min_{<_o} S(A,\Sigma)\right)^t .$$

□

6.3 Proof that removing A is a contraction on $G(\Sigma + A)$

Definition 24. For a multiconnected split set Σ and an ordered split $A \in \check{\Sigma}$, let $L(A,\Sigma) \equiv \max_{\downarrow}(\min_{<_o} S(A,\Sigma))^t$. If $A \notin \Sigma$ but $A \in NW(\Sigma + A)$ then $L(A,\Sigma) \subseteq NW(\Sigma + A)$ since $L(A, \Sigma + A) \subset \nu(A, \Sigma + A)$ by Theorem 23. Since $L(A, \Sigma + A)$ does not contain A we note that $L(A,\Sigma) = L(A, \Sigma + A)$. Since elements in $NW(\Sigma + A)$ minus A and A^t are also in $NW(\Sigma)$, we also have $L(A,\Sigma) \subseteq NW(\Sigma)$ by Lemma 27.

Lemma 25. *If Σ is a multiconnected split set andA is informative, then $S(A,\Sigma)$ is not empty.*

Proof. Let x be a member of A_2 and let $C = \mathcal{L} - x|x$. Now, $C \in \check{\Sigma}$ since Σ must contain all leaf splits of the leaf set $\mathcal{L}$. We have $A_1 \subset C_1$ because $A_1 \subset C_1 \cup C_2$ and $A_1 \cap C_2 = A_1 \cap \{x\} = \emptyset$. Clearly $C_2 \subset A_2$. Therefore $A < C$, and so $C \in S(A,\Sigma)$. □

Lemma 26. *If Σ is a multiconnected split set and $A \in NW(\check{\Sigma})$ and then $|\nu(A,\Sigma)| = 1$ and $L(A,\Sigma)$ is empty if $|A_2| = 1$, whereas $|\nu(A,\Sigma)| > 1$* and $L(A,\Sigma)$ is non-empty if $|A_2| > 1$.

Proof. If $|A_2| = 1$, then let us assume that there exists a $B \in \nu(A,\Sigma)$ such that $B \neq A$. Since $A \neq B$, we must have that $A \ll B^t$ in Σ. However, $A < B^t$ would require $B_1 \subset A_2$, which would require $|B_1| = 0$. This cannot happen, because then B would not be a split. Therefore, if $|A_2| = 1$, then there are no other members of $\nu(A,\Sigma)$, besides A and $|\nu(A,\Sigma)| = 1$. Since $\nu(A,\Sigma)$ contains $L(A,\Sigma)$ but $L(A,\Sigma)$ does not contain A, then $L(A,\Sigma)$ must be empty.

If $|A_2| > 1$, then $S(A,\Sigma)$ is not empty by Lemma 25, the set $L(A,\Sigma) = \max_{\downarrow}(\min_{<_o} S(A,\Sigma))^t$ cannot be empty either. Therefore it contains some element D, and thus by Theorem 23 $D \in \nu(A,\Sigma)$ and $D \neq A$. Therefore $|\nu(A,\Sigma)|$ is at least 2. □

Lemma 27. *If Σ and $\Sigma + A$ are multiconnected split sets and $\{A, A^t\} \subseteq NW(\Sigma + A)$ and $B \in NW(\Sigma + A) \cap \check{\Sigma}$ then $B \in NW(\Sigma)$. That is, $NW(\Sigma + A) \cap \check{\Sigma} \subseteq NW(\Sigma)$, or $NW(\Sigma + A) \subseteq NW(\Sigma) + \{A, A^t\}$. Furthermore, $NW(\Sigma + A) \cap \{A, A^t\}^C = NW(\Sigma + A) \cap NW(\Sigma)$.*

Proof. Let us assume by way of contradiction that there exists some ordered split $C \in \check{\Sigma}$ such that $B \Downarrow C$ in Σ. We note that $B \not\Downarrow C$ in $\Sigma + A$ since $B \in NW(\Sigma + A)$. Therefore, we can only have $B \Downarrow C$ in Σ if $B < A \downarrow C$ or $B < A^t \downarrow C$, since A and A^t are the only elements in $\Sigma \check{+} A$ that are not in $\check{\Sigma}$, and none of the elements in $\check{\Sigma}$ can be between B and C in this way. But then there can be no element $D \in \Sigma + A$ such that $B < A < D \downarrow C$ or $B < A^t < D \downarrow C$, since this would contradict $B \Downarrow C$ in Σ. Therefore either $A \Downarrow C$ or $A^t \Downarrow C$ in $\Sigma + A$. But this contradicts our assumptions, and therefore there can be no C such that $B \Downarrow C \in \Sigma$. Therefore $B \in NW(\Sigma)$, and $NW(\Sigma + A) \cap \check{\Sigma} \subseteq NW(\Sigma)$.

Since $NW(\Sigma + A) \cap \check{\Sigma} = NW(\Sigma + A) \cap \left\{A, A^t\right\}^C$, we have

$$\begin{aligned} NW(\Sigma + A) \cap \left\{A, A^t\right\}^C &= NW(\Sigma + A) \cap \left\{A, A^t\right\}^C \cap NW(\Sigma) \\ &= NW(\Sigma + A) \cap NW(\Sigma). \end{aligned}$$

□

Definition 28. We define the set of informative splits in a split set Σ as $I(\Sigma) \equiv \{\sigma \in \Sigma : \sigma \text{ is informative}\}$.

Lemma. *If Σ and $\Sigma + A$ are multiconnected split sets and $\left\{A, A^t\right\} \subseteq NW(\Sigma + A)$ and $B \in NW(\Sigma)$ then either $B \in NW(\Sigma + A)$ or $B \Downarrow A$.*

Proof. Consider some $B \in NW(\Sigma)$. Let us assume that $B \notin NW(\Sigma + A)$. Then there exists some $C \in \Sigma \check{+} A$ such that $B \Downarrow C$ in $\Sigma + A$. But $B \not\Downarrow C$ in Σ since $B \in NW(\Sigma)$. Now, suppose that $C \notin \check{\Sigma}$, then C is A or A^t and $B \Downarrow A$ in $\Sigma + A$. Alternatively, if $C \in \check{\Sigma}$, then $B \downarrow C$. There cannot be any $D \in \check{\Sigma}$ such that $B < D \downarrow C$, because then $D \in \Sigma \check{+} A$ and so if there were such a D then $B \not\Downarrow C$ in $\Sigma + A$. But then $B \Downarrow C$ in Σ, which contradicts $B \in NW(\Sigma)$. Therefore, only the $C \notin \check{\Sigma}$ case is allowed, and so our assumption $B \notin NW(\Sigma + A)$ implies $B \Downarrow A$.

□

Lemma 29. *If $(B, C) \in N(\Sigma + A) \cap \hat{\Sigma}$ then $(B, C) \in N(\Sigma)$.*

Proof. According to Lemma 27, $\{B, C\} \subseteq NW(\check{\Sigma})$ since $\{B, C\} \subseteq NW(\Sigma \check{+} A) \cap \hat{\Sigma}$. If there is no $D \in \Sigma \check{+} A$ such that $B < D < C^t$ then there isn't going to be one in $\check{\Sigma}$. Therefore $(B, C) \in N(\Sigma)$. □

Lemma 30. *If Σ and $\Sigma + \check{A}$ are multiconnected trees and $\left\{A, A^t\right\} \subseteq NW(\Sigma + A)$ and $B \in \check{\Sigma}$ and $(A, B) \in N(\Sigma + A)$ then $(A, B) \in N(\Sigma)$*

Proof. A and C are in $NW(\check{\Sigma})$ since they are in $NW(\Sigma + A)$ by Lemma 27. Now since there is no $D \in \Sigma \check{+} A$ such that $A < D < B^t$ then there is also no such D in $\check{\Sigma}$. Then $A \ll B^t$ in Σ and so $(A, B) \in N(\Sigma)$. □

Definition 31. Let $J_1(A, \Sigma)$ be $L(A, \Sigma) \cup L(A^t, \Sigma)$. Thus $J_1(A, \Sigma) = J_1(A, \Sigma + A)$. We note that also $J_1(A, \Sigma) = \left[\nu(A, \Sigma + A) \cup \nu(A^t, \Sigma + A)\right] \cap \check{\Sigma}$. We also note that $J_1(A, \Sigma) \subseteq NW(\Sigma)$ and $J_1(A, \Sigma) \subseteq NW(\Sigma + A)$.

Definition 32. Let $K(A, \Sigma)$ be $NW(\Sigma + A) \cap \left\{\nu(A, \Sigma) \cup \nu(A^t, \Sigma)\right\}^C$. Since $\nu(A, \Sigma) \cup \nu(A^t, \Sigma) \subseteq NW(\Sigma + A)$ we have the following as a partion of $NW(\Sigma + A)$:

$$\begin{aligned} NW(\Sigma + A) &= \nu(A, \Sigma) \cup \nu(A^t, \Sigma) \cup \left[NW(\Sigma + A) \cap \left(\nu(A, \Sigma) \cup \nu(A^t, \Sigma)\right)^C\right] \\ &= \nu(A, \Sigma) \cup \nu(A^t, \Sigma) \cup K(A, \Sigma). \end{aligned}$$

Now since any split B with $A \in \nu(B, \Sigma + A)$ also has $B \in \nu(A, \Sigma + A)$, note that $K(A, \Sigma)$ is the set of splits B in $NW(\Sigma + A)$ such that $\nu(B, \Sigma + A)$ does not contain A or A^t. Also note that

$$\begin{aligned} K(A, \Sigma) &= NW(\Sigma + A) \cap \left\{J_1(A, \Sigma) \cup A \cup A^t\right\}^C \\ &= NW(\Sigma + A) \cap J_1(A, \Sigma)^C \cap \left\{A \cup A^t\right\}^C. \end{aligned}$$

By Lemma 27 we have $NW(\Sigma + A) \cap \{A \cup A^t\}^C = NW(\Sigma + A) \cap NW(\Sigma)$, so that

$$K(A, \Sigma) \quad = \quad NW(\Sigma) \cap NW(\Sigma + A) \cap J_1^C(A, \Sigma).$$

Lemma 33. *If Σ and $\Sigma + A$ are multiconnected split sets and $\{A, A^t\} \subseteq NW(\Sigma + A)$, then for any split $B \in K(A, \Sigma)$ we have that $\nu(B, \Sigma + A) = \nu(B, \Sigma)$.*

Proof. For any $B \in K(A, \Sigma)$, Theorem 23 implies that $\nu(B, \Sigma{+}A) = \{B\} \cup \max_{\downarrow} (\min_{<_o} S(B, \Sigma + A))^t$. Assume by way of contradiction that there is some $B \in K(A, \Sigma)$ such that $\min_{<_o} S(B, \Sigma + A)$ contains A or A^t. Then Lemma 18 implies that $A \in \nu(B, \Sigma + A)$ or $A^t \in \nu(B, \Sigma + A)$ because A and A^t are both in $NW(\Sigma + A)$. This then implies that $B \in \nu(A, \Sigma + A) \cup \nu(A^t, \Sigma + A)$ because the sets $\nu(\cdot, \Sigma + A)$ are equivalence classes. But this contradicts the premise that $B \in K(A, \Sigma)$, since $K(A, \Sigma)$ has no overlap with $\nu(A, \Sigma + A) \cup \nu(A^t, \Sigma + A)$. Therefore, $\min_{<_o} S(B, \Sigma + A)$ does not contain A or A^t.

Since $S(B, \Sigma + A)$ does not contain A or A^t for any $B \in K(A, \Sigma)$, then for any such B we have

$$\min_{<_o} S(B, \Sigma + A) \quad = \quad \min_{<_o} S(B, \Sigma),$$

since removing any element from a set cannot make a minimal element stop being minimal. Therefore

$$\max_{\downarrow} \left(\min_{<_o} S(B, \Sigma + A) \right)^t \quad = \quad \max_{\downarrow} \left(\min_{<_o} S(B, \Sigma) \right)^t$$

and so $\nu(B, \Sigma + A) = \nu(B, \Sigma)$. $\square$

Lemma 34. *If Σ and $\Sigma + B$ are multiconnected trees and $\{B, B^t\} \subseteq NW(\Sigma + B)$ and $A, B, C \in \check{\Sigma}$ and $A \ll B \ll C$ in $\Sigma + B$, then $A \ll C$ in Σ.*

Proof. A and C^t are in $NW(\check{\Sigma})$ since they are in $NW(\Sigma + B)$ by Lemma 27.

Suppose $A \not\ll C$ in Σ. Then there exists some $D \in \check{\Sigma}$ such that $A < D < C$. Now, if $A < B < C$ and $A < D < C$ then either $B < D$ or $D < B$ by Lemma 81. Now, if $B < D$ then $A < B < D < C$, which contradicts $B \ll C$ in $\Sigma + B$. Alternatively, if $D < B$, then $A < D < B < C$, which contradicts $A \ll B$ in $\Sigma + B$. Therefore, no such D exists, and $A \ll C$ in Σ. $\square$

Lemma 35. *If Σ and $\Sigma + A$ are multiconnected trees and $A \in I(\Sigma + A)$, then the set $J_1(A, \Sigma)$ is a subset of $\check{\Sigma}$, is non-empty, and every member is equivalent under $N(\Sigma)$. Therefore there is a single node $J(A, \Sigma)$ in $V(\Sigma)$ which contains all of $J_1(A, \Sigma)$.*

Proof. Both of $S(A, \Sigma + A)$ and $S(A^t, \Sigma + A)$ cannot contain either A or A^t, and thus are subsets of $\check{\Sigma}$. Therefore $J_1(A, \Sigma) \subseteq \check{\Sigma}$. Since A is informative, neither of $S(A, \Sigma + A)$ or $S(A^t, \Sigma + A)$ can be empty by Lemma 25. Therefore, $L(A^t, \Sigma)$ and $L(A, \Sigma)$ are non-empty, and $J_1(A, \Sigma)$ is non-empty also.

Consider two elements B and C of $J_1(A, \Sigma)$. If $\{B, C\} \subseteq L(A, \Sigma)$ or $\{B, C\} \subseteq L(A^t, \Sigma)$ then $(B, C) \in N(\Sigma + A)$ and so $(B, C) \in N(\Sigma)$ by Lemma 30. If $B \in L(A^t, \Sigma)$ and $C \in L(A, \Sigma)$, then $B \ll A \ll C^t$ in $\Sigma + A$. Then $A \ll C^t$ in Σ by Lemma 34 and so $(B, C) \in N(\Sigma)$. The same is true by symmetry of A and A^t if $C \in L(A^t, \Sigma)$ and $B \in L(A, \Sigma)$. Therefore, for any $B, C \in J_1(A, \Sigma)$ we have $B \sim_{N(\Sigma)} C$ and so there is a single node which contains $J_1(A, \Sigma)$. We term this node $J(A, \Sigma)$. $\square$

Definition 36. Let $J(A, \Sigma)$ be the node in $V(\Sigma)$ such that contains all of $J_1(A, \Sigma)$. We know that there is such a node by Lemma 35. For each $C \in J_1(A, \Sigma)$, C is also in $\check{\Sigma}$ and $J(A, \Sigma) = \nu(C, \Sigma)$.

Definition 37. Let $J_2(A,\Sigma)$ be the set $NW(\Sigma) \cap NW(\Sigma + A)^C$.

Lemma 38. *If Σ and $\Sigma + \check{A}$ are multiconnected trees and $\{A, A^t\} \subseteq NW(\Sigma + A)$ then $J_2(A,\Sigma) \subseteq J(A,\Sigma)$.*

Proof. Now if $B \in J_2(A,\Sigma)$ then $B \Downarrow A$ in $\Sigma + A$ by Lemma 6.3. Now $L(A^t, \Sigma + A)$ is not empty, and so there exists some $C^t \in L(A^t, \Sigma + A)$ and $\nu(C^t, \Sigma) = J(A,\Sigma)$ by Lemma 35. We note that $A^t \ll C^t$ in $\Sigma + A$ and so $C \ll A$ in $\Sigma + A$. By Lemma 75, either $C < B^t$ or $B \Downarrow C$ in $\Sigma + A$. But if $B \Downarrow C$ in $\Sigma + A$, then $B \Downarrow C$ in Σ, which contradicts $B \in NW(\Sigma)$. Therefore $C < B^t$.

Now, $C < A$ and $B \Downarrow A$ and $C < B^t$. Suppose there were some $D \in \check{\Sigma}$ such that $C < D < B^t$. Then by Lemma 90, we must have either $C < D < A$ or $B < D^t \downarrow A$. Now, neither of these can be true, since $C < D < A$ contradicts $C \ll A$ in $\Sigma + A$ and $B < D^t \downarrow A$ contradicts $B \Downarrow A$ in $\Sigma + A$. Therefore there can be no $D \in \check{\Sigma}$ such that $C < D < B^t$. And since C and B are in $NW(\Sigma)$ we must have $C \ll B^t$ in Σ. But then $B \in \nu(C,\Sigma)$, and $\nu(C,\Sigma) = J(A,\Sigma)$.

Therefore if any $B \in J_2(A,\Sigma)$ then also $B \in J(A,\Sigma)$ and $J_2(A,\Sigma) \subseteq J(A,\Sigma)$. □

Lemma 39. *If Σ and $\Sigma + A$ are multiconnected trees and $\{A, A^t\} \subseteq NW(\Sigma + A)$, then $NW(\Sigma)$ can be partitioned into the non-overlapping subsets $J_1(A,\Sigma)$, $J_2(A,\Sigma)$, and $K(A,\Sigma)$.*

Proof. We first note that splits in $NW(\Sigma)$ can be partitioned into $NW(\Sigma) \cap NW(\Sigma + A)$ and $J_2(A,\Sigma)$, and that these two sets do not overlap:

$$\begin{aligned} NW(\Sigma) &= [NW(\Sigma) \cap NW(\Sigma + A)] \cup \left[NW(\Sigma) \cap NW(\Sigma + A)^C\right] \\ &= [NW(\Sigma) \cap NW(\Sigma + A)] \cup J_2(A,\Sigma) \end{aligned}$$

Since $J_1(A,\Sigma) \subseteq NW(A,\Sigma)$, we can also partition $NW(\Sigma)$ into $J_1(A,\Sigma)$ and $NW(\Sigma) \cap J_1(A,\Sigma)^C$:

$$\begin{aligned} NW(\Sigma) &= [NW(\Sigma) \cap J_1(A,\Sigma)] \cup \left[NW(\Sigma) \cap J_1(A,\Sigma)^C\right] \\ &= J_1(A,\Sigma) \cup \left[NW(\Sigma) \cap J_1(A,\Sigma)^C\right]. \end{aligned}$$

Using this result, we can further partition $NW(\Sigma) \cap NW(\Sigma + A)$ into $J_1(A,\Sigma) \cap NW(\Sigma + A)$ and $K(A,\Sigma)$:

$$\begin{aligned} NW(\Sigma) \cap NW(\Sigma + A) &= [J_1(A,\Sigma) \cap NW(\Sigma + A)] \cup \left[NW(\Sigma) \cap J_1(A,\Sigma)^C \cap NW(\Sigma + A)\right] \\ &= [J_1(A,\Sigma) \cap NW(\Sigma + A)] \cup K(A,\Sigma). \end{aligned}$$

Since $J_1(A,\Sigma) \subseteq NW(A, \Sigma + A)$ as well, we can simplify the partition of $NW(\Sigma) \cap NW(\Sigma + A)$ into just $J_1(A,\Sigma)$ and $K(A,\Sigma)$.

$$NW(\Sigma) \cap NW(\Sigma + A) = J_1(A,\Sigma) \cup K(A,\Sigma).$$

Thus, by partitioning $NW(\Sigma) \cap NW(\Sigma + A)$, we yield a final partition of $NW(\Sigma)$ into non-overlapping subsets $J_1(A,\Sigma)$, $J_2(A,\Sigma)$, and $K(A,\Sigma)$. □

Lemma 40. *If Σ and $\Sigma + A$ are multiconnected trees and $\{A, A^t\} \subseteq NW(\Sigma + A)$, then the vertices $V(\Sigma)$ are all members of $V(\Sigma + A)$, except for $J(A,\Sigma)$.*

Proof. By Lemma 39, we can express the nodes of Σ as:

$$\begin{aligned} V(\Sigma) &= \bigcup_{B\in NW(\Sigma)} \nu(B,\Sigma) \\ &= \left(\bigcup_{B\in J_1(A,\Sigma)} \nu(B,\Sigma)\right) \cup \left(\bigcup_{B\in K(A,\Sigma)} \nu(B,\Sigma)\right) \cup \left(\bigcup_{J_2(A,\Sigma)} \nu(B,\Sigma)\right). \end{aligned}$$

Now, by Lemma 35, any $B \in J_1(A,\Sigma)$ has $\nu(B,\Sigma) = J(A,\Sigma)$. Furthermore, by Lemma 38, any $B \in J_2(A,\Sigma)$ also has $\nu(B,\Sigma) = J(A,\Sigma)$. Finally, we note that $J_1(A,\Sigma)$ cannot be empty, because $J_1(A,\Sigma)$ contains $L(A,\Sigma)$ which cannot be empty because A is an informative split **FIXME**. Therefore:

$$\begin{aligned} V(\Sigma) &= \left(\bigcup_{B\in J_1(A,\Sigma)} J(A,\Sigma)\right) \cup \left(\bigcup_{B\in K(A,\Sigma)} \nu(B,\Sigma)\right) \cup \left(\bigcup_{B\in J_2(A,\Sigma)} J(A,\Sigma)\right) \\ &= J(A,\Sigma) \cup \left(\bigcup_{B\in K(A,\Sigma)} \nu(B,\Sigma)\right). \end{aligned}$$

Now, by Lemma 33 $\nu(B,\Sigma) = \nu(B,\Sigma+A)$ for every node $B \in K(A,\Sigma)$. Therefore

$$\begin{aligned} V(\Sigma) &= J(A,\Sigma) \cup \left(\bigcup_{B\in K(A,\Sigma)} \nu(B,\Sigma+A)\right) \\ V(\Sigma) - J(A,\Sigma) &\subseteq V(\Sigma+A). \end{aligned}$$

□

Lemma 41. *If Σ and $\Sigma+A$ are multiconnected trees and $\{A, A^t\} \subseteq NW(\Sigma+A)$ and $A \in I(\Sigma+A)$ then $J(A,\Sigma)$ is not in $V(\Sigma+A)$*

Proof. First we note that $L(A,\Sigma)$ and $L(A^t,\Sigma)$ are both non-empty, since A is informative in $\Sigma+A$ by Lemma 26. Therefore there is some member $B \in L(A,\Sigma)$ and some member $C \in L(A^t,\Sigma)$. By the definition of $J(A,\Sigma)$, both B and C are members of $J(A,\Sigma)$.

However, we also have that B is equivalent to A and C is equivalent to A^t under $N(\Sigma+A)$ by Theorem 23. But A and A^t cannot be equivalent to each other under $N(\Sigma+A)$ since it is not the case that $A = A^t$ or $A < \left(A^t\right)^t$. Therefore, B and C cannot be equivalent to each other under $N(\Sigma+A)$. Therefore $J(A,\Sigma)$ is not an equivalence class of $N(\Sigma+A)$, and so $J(A,\Sigma)$ is not in $V(\Sigma+A)$, since $V(\Sigma+A)$ contains only equivalence classes under $N(\Sigma+A)$. □

Lemma 42. *If Σ and $\Sigma+A$ are multiconnected trees and $\{A, A^t\} \subseteq NW(\Sigma+A)$ and $A \in I(\Sigma+A)$ then the move from $G(\Sigma+A)$ to $G(\Sigma)$ removes $\nu(A,\Sigma+A)$ and $\nu(A^t,\Sigma+A)$, adds $J(A,\Sigma)$ and otherwise leaves the vertices $V(\Sigma+A)$ unchanged.*

Proof. Any node in $V(\Sigma+A)$ that is not $\nu(A)$ or $\nu(A^t)$ is also in $V(\Sigma)$ by Lemma 33, and any node that in $V(\Sigma)$ that is not $J(A,\Sigma)$ is also in $V(\Sigma+A)$ by Lemma 40. The nodes A and A^t are not in $V(\Sigma)$ because A and A^t are not in Σ and therefore not in $NW(\Sigma)$. The node $J(A,\Sigma)$ is not in $V(\Sigma+A)$ by Lemma 41. □

Lemma. *If $B \Downarrow A$ in $\Sigma + A$ and $B \notin NW(\Sigma)$ and B does not directly wander over any split in $L(A, \Sigma) \cup L(A^t, \Sigma)$ then B must wander over some split in $J_2(A, \Sigma)$.*

Proof. Because $B \notin NW(\Sigma)$ there must be some $C \in NW(\Sigma)$ such that $B \Downarrow C$. There are three cases:

- C or C^t is in $S(A, \Sigma)$ or $S(A^t, \Sigma)$

  If $C \in S(A, \Sigma)$ then there must exist some $M \in \min_{<_o} S(A, \Sigma)$ such that $A < M$ and either either $C = M$ or $M < C$ or $M \downarrow C$. We must have either $B \Downarrow M$ or $B < M$ by Lemma 75. Now, if $C = M$, clearly it is not the case that $B < M$ since this would contradict $B \downarrow C$. If $M < C$, then it is not the case that $B < M$ because $B < M < C$ would contradict $B \Downarrow C$. If $M \downarrow C$ then $B < M \downarrow C$ would also contradict $B \Downarrow C$. Hence we have $B \Downarrow M$.
  Now, by Lemma 22, either $M^t \in NW(\Sigma)$ or $M^t \Downarrow M_2 \in \min_{<_o} S(A, \Sigma)$. By repeatedly applying this fact we find some element M_3 of $\min_{<_o} S(A, \Sigma)$ such that $M_3^t \in NW(\Sigma)$ and $B \Downarrow M_3$. Thus, we cannot have C or C^t in $S(A, \Sigma)$ or $S(A^t, \Sigma)$ because this would contradict our assumption that B does not directly wander over any split in $L(A, \Sigma)$.

  go back and augment/clean-up previous proofs about this?

- $A \perp C$

  There must be some $F \in \Sigma$ such that $F \downarrow A$ and $F^t \downarrow C$. Suppose that there is some Z such that $B < Z \downarrow F$. Then $B < Z \downarrow C$, which contradicts $B \Downarrow C$. Thus we must have $B \Downarrow F$. But then $B \Downarrow F$ and $F \downarrow A$, for which case see below.

- $C \downarrow A$ or $C^t \downarrow A$

  Suppose $C \downarrow A$. Consider the set $\{C\} \cup \{Z : C < Z \downarrow A\} \cup \{Z : C \downarrow Z \downarrow A\}$. B wanders directly over each element of $\{Z : C < Z \downarrow A\}$ by Lemma 92, and over each element of $\{Z : C \downarrow Z \downarrow A\}$ by Lemma 91. Since B wanders directly over C as well, B directly wanders over every element of this set.

  Now, take the maximal element M of the set under the order $<_o$. Clearly $M \downarrow A$. Suppose that there exists some $D \in \check{\Sigma}$ such that $M < D \downarrow A$. Such a D must be in the set above since $C \leq_o M < D \downarrow A$ to $C \leq_o D \downarrow A$. But then such a D would be more minimal than M, which is a contradiction. Therefore, there is no such D and $M \Downarrow A$.

  Now suppose that there is some element $E \in \check{\Sigma}$ such that $M \Downarrow E$ and $M \Downarrow A$. Such an E cannot be in $S(A, \Sigma)$ or $S(A^t, \Sigma)$ because this would imply that M wanders directly over some element of $L(A, \Sigma)$ or $L(A^t, \Sigma)$ by the first case of this theorem. If such an E wandered over A then we would have $C \leq_o M \downarrow E \downarrow A$, which would put E is the set above. This would contradict the maximality of M, so E could not wander over A. Finally, we could have $E \perp A$. This would require that there is some $F \downarrow A$ and $F^t \downarrow E$. But then $M \downarrow F \downarrow A$, which contradicts the maximality of M. We therefore conclude that there can be no such E. Therefore, M directly wanders over A and A alone, and so M is in $J_2(A, \Sigma)$. Therefore B wanders over an element of $J_2(A, \Sigma)$.

□

Lemma 43. *Each edge $e(B, \Sigma)$ in $E(\Sigma)$ is a transformed version of edge $e(B, \Sigma+A)$ where the nodes $\nu(A, \Sigma)$ and $\nu(A^t, \Sigma)$ have been replaced with $J(A, \Sigma)$, and (ii) other nodes have been unchanged. Furthermore, only the edge $e(A, \Sigma + A)$ in $E(\Sigma + A)$ doesn't correspond to an edge in $e(\Sigma)$ in this way.*

Proof. It suffices to show that $T(B,\Sigma)$ can be created from $T(B,\Sigma+A)$ by replacing the nodes $\nu(A,\Sigma)$ and $\nu(A^t,\Sigma)$ with $J(A,\Sigma)$. This is because (i) Σ already has A removed from $\Sigma+A$. (ii) the correspondence $e(B,\Sigma)\sim e(B,\Sigma+A)$ is already known and one-to-one. (?)

Let us first consider B in $NW(\Sigma)$.

- If $B\in L(A,\Sigma)$ then $T(B,\Sigma+A)=\nu(A,\Sigma+A)$ and $T(B,\Sigma)=J(A,\Sigma)$. OK.
- If $B\in L(A^t,\Sigma)$ then $T(B,\Sigma+A)=\nu(A^t,\Sigma+A)$ and $T(B,\Sigma)=J(A,\Sigma)$. OK.
- If $B\in K(A,\Sigma)$ then B is in $NW(\Sigma)$ and $NW(\Sigma+A)$, and $\nu(B,\Sigma)=\nu(B,\Sigma+A)$. Hence $T(B,\Sigma)=\nu(B,\Sigma)=\nu(B,\Sigma+A)=T(B,\Sigma+A)$. Since this single node is neither $\nu(A,\Sigma)$ nor $\nu(A,\Sigma+A)$, this is OK.
- If $B\in J_2(A,\Sigma)$ then B is in $NW(\Sigma)$ but not in $NW(\Sigma{+}A)$. $T(B,\Sigma{+}A)=\big\{\nu(A,\Sigma+A),\nu(A^t,\Sigma+A)\big\}$ and $T(B,\Sigma)=J(A,\Sigma)$. OK.

Thus, if $B\in NW(\Sigma)$ then $T(B,\Sigma)$ is a transformed version of $T(B,\Sigma+A)$.

Secondly, let us consider $B\notin NW(\Sigma)$. In this case, B is not in $NW(\Sigma+A)$ either, and so the expression for $T(B,\Sigma+A)$ is:

$$\begin{aligned}
T(B,\Sigma+A) &= \bigcup_{C\in NW(\Sigma+A),B\Downarrow C} \nu(C,\Sigma+A)\\
&= \bigcup_{C\in L(A,\Sigma),B\Downarrow C} \nu(C,\Sigma+A)\cup \bigcup_{C\in\{A,A^t\},B\Downarrow C} \nu(C,\Sigma+A)\cup \bigcup_{C\in L(A^t,\Sigma),B\Downarrow C} \nu(C,\Sigma+A)\cup \bigcup_{C\in K(A,\Sigma),B\Downarrow}\\
&= \bigcup_{C\in L(A,\Sigma)\cup\{A\},B\Downarrow C} \nu(A,\Sigma+A)\cup \bigcup_{C\in L(A^t,\Sigma)\cup\{A^t\},B\Downarrow C} \nu(A^t,\Sigma+A)\cup \bigcup_{C\in K(A,\Sigma),B\Downarrow C} \nu(C,\Sigma)
\end{aligned}$$

Likewise, the expression for $T(B,\Sigma)$ is:

$$\begin{aligned}
T(B,\Sigma) &= \bigcup_{C\in NW(\Sigma),B\Downarrow C} \nu(C,\Sigma)\\
&= \bigcup_{C\in L(A,\Sigma),B\Downarrow C} \nu(C,\Sigma)\cup \bigcup_{C\in L(A^t,\Sigma),B\Downarrow C} \nu(C,\Sigma)\cup \bigcup_{C\in J_2(A,\Sigma),B\Downarrow C} \nu(C,\Sigma)\cup \bigcup_{C\in K(A,\Sigma),B\Downarrow C} \nu(C,\Sigma)\\
&= \bigcup_{C\in L(A,\Sigma),B\Downarrow C} J(A,\Sigma)\cup \bigcup_{C\in L(A^t,\Sigma),B\Downarrow C} J(A,\Sigma)\cup \bigcup_{C\in J_2(A,\Sigma),B\Downarrow C} J(A,\Sigma)\cup \bigcup_{C\in K(A,\Sigma),B\Downarrow C} \nu(C,\Sigma).
\end{aligned}$$

Since $\nu(C,\Sigma)$ will never equal $J(A,\Sigma)$ when $C\in K(A,\Sigma)$, we see that $T(B,\Sigma)$ only contains $J(A,\Sigma)$ when B wanders directly over splits in $L(A,\Sigma)\cup L(A^t)\cup J_2(A,\Sigma)=J(A,\Sigma)$. In all of these cases, B also directly wanders over a split in $\nu(A,\Sigma)\cup\nu(A^t,\Sigma)$. Clearly this is true when B wanders over a split in $L(A,\Sigma)\cup L(A^t,\Sigma)$, since $L(A,\Sigma)\cup L(A^t)\subseteq\nu(A,\Sigma)\cup\nu(A^t,\Sigma)$. However, it is also true when B wanders over a split in $J_2(A,\Sigma)$. This is because if $B\Downarrow C\in J_2(A,\Sigma)$ then $B\Downarrow A$ since $C\Downarrow A$ by Lemma 94. Therefore, $T(B,\Sigma+A)$ contains $\nu(A,\Sigma+A)$ or $\nu(A^t,\Sigma+A)$ if $T(B,\Sigma)$ contains $J(A,\Sigma)$.

We also seek to show that if B wanders directly over a split in $\nu(A,\Sigma)\cup\nu(A^t,\Sigma)$ then B wanders directly over a split in $J(A,\Sigma)$. If B wanders directly over a split in $L(A,\Sigma)\cup L(A^t,\Sigma)$ then the conclusion follows. If B does not wander over any split in $L(A,\Sigma)\cup L(A^t,\Sigma)$, then by Lemma 6.3 B must wander over some split in $J_2(A,\Sigma)$. Therefore, if B wanders over a split in $\nu(A,\Sigma+A)\cup\nu(A^t,\Sigma+A)$ in $\Sigma+A$, then B wanders over a split of $J(A,\Sigma)$ in Σ. Therefore, $T(B,\Sigma)$ contains $J(A,\Sigma)$ if $T(B,\Sigma+A)$ contains $\nu(A,\Sigma+A)$ or $\nu(A^t,\Sigma+A)$.

Therefore, $T(B,\Sigma)$ contains $J(A,\Sigma)$ if and only if $T(B,\Sigma+A)$ contains $\nu(A,\Sigma+A)$ or $\nu(A^t,\Sigma+A)$. □

6.4 Proofs: embedded graphs are trees, and the edge $e(A)$ implies A in $G(\Sigma+A)$.

Theorem 44. *All embedded graphs are trees (in progress)*

1. *The two endpoints of a doubly non-wandering edge A must be distinct.* ($\nu(A) \neq \nu(A^t)$)
2. *Removing such an edge A from $\Sigma + A$ therefore removes exactly 1 node and 1 edge from $G(\Sigma + A) \to G(\Sigma)$*
3. *The number of edges and vertices in an embedded graph of $G(\Sigma)$ is identical to the number in $G(\Sigma)$.*
4. *Removing A from $\Sigma + A$ performs a contraction on each embedded graph of $G(\Sigma + A)$, and by doing so yields all the embedded graphs of $G(\Sigma)$. This means that the each embedded graph of $G(\Sigma)$ has 1E, 1V, and 0components less than some embedded graph of $G(\Sigma)$. And every embedded graph of $G(\Sigma + A)$ has 1E, 1V, and 0components more than some embedded graph of $G(\Sigma)$.*
5. *There will always be a doubly-non-wandering internal edge, if there are any internal edges.*
6. *Therefore, we can get down to* 0 *internal edges (in the embedded graphs and in $G(\Sigma)$), in increments of 1E, 1V.*
7. *And then there will be only 1 embedded graph with $|\mathcal{L}|$ edges, $|\mathcal{L}|+1$ vertices, and 1 component.*
8. *Therefore, every embedded graph in $G(\Sigma)$ with B internal splits has $|L+B|$ edges, $|L|+B+1$ vertices, and 1 component.*
9. *Therefore every embedded graph in $G(\Sigma)$ is a tree.*

1. Why did I have a problem with the old proof?

Theorem 45. *Each edge $e(A) = (T(A^t), T(A))$ induces the split A. That is, when this edge is cut, then every $x \in A$ is inserted into some single (non-wandering) vertex $J(A,\Sigma)$. (in progress)*

1. *Let us assume that $G(\Sigma)$ is in fact a multiconnected* tree *(e.g. all embedded graphs are trees) so that every edge in $\min_{<_0} S(A,\Sigma)$ induces its split* ***and the split of all relevant internal nodes****.*
2. *Prove a decomposition: that all leaf charaters in A_1 are connected to the endpoints of (exactly) one element of $\min_{<_0} S(A,\Sigma)$.*
3. *Then prove that the edges that wander must choose endpoints that are behind the elements in $D(A,\Sigma)$ that do* not *wander.*

1. How does the set of embedded trees in an MC graph (in general) relate to the splits of that graph?
   (a) prove: every edge in an embedded graph (the multi-edges version) induces a split that implies the split of its parent edge.

(b) proof: if the edge (u,v) induces the split $A_{v,u,G}|A_{u,v,G}$ when cut, then $A_{u,v,G}$ is the set of nodes that attach the v but not u in all embedded graphs when the edge (u,v) is cut in the embedded graph. Now, choosing an attachment point reduces the number of embedded graphs. So, the nodes that attach to v but not u in *some* embedded graphs must be at least as large as the number that attaches to v but not u in *all* of the embedded graphs.

(c) In other words, $A_{u,v,G} = \bigcap_{g\in G} A_{u,v,g}$. This set can only increase as G decreases.

6.5 Defining resolution and embedded graphs

6.5.1 Definition of the graph

Points, edges, and target set A multiconnected graph $\mathcal{G}$ is a collection (V,E,T) of vertex set V, edge multiset E, and vertex target sets T. Each vertex $v \in V$ has a corresponding set of vertices $T(\mathcal{G},v) \subset V$ that we refer to as the targets of v. $T(\mathcal{G},v)$ must not be empty. If $T(\mathcal{G},v) \neq \{v\}$, then we say that v is a wandering vertex. In this case, the targets of v are also referred to as the attachment points of v, and we say that v wanders over them. Only vertices with degree 1 are allowed to wander, so that in effect edges wander, but vertices in general cannot be arbitrarily set to be equal. We consider a (standard) graph (V,E) to be equivalent to the multiconnected graph (V,E,I) where I is the identity map so that all vertices are non-wandering.

No cycles in transitive closure of attachment points We refer to the wandering vertices of $\mathcal{G}$ as $\overrightarrow{V}(\mathcal{G})$, and the other vertices of G as $\dot{V}(\mathcal{G})$, so that $V(\mathcal{G}) = \overrightarrow{V}(\mathcal{G}) \cup \dot{V}(\mathcal{G})$. More interestingly, consider the transitive closure of the relation "v wanders over w". This yields the set $T^+(\mathcal{G},v)$ of vertices that v could be identified with through a *sequence* of 1 or more attachments. We say that v is above w if $w \in T^+(\mathcal{G},v)$ and $w \neq v$.

The last restriction on $\mathcal{G}$ is that $T^+(\mathcal{G},v)$ must not contain v unless $T^+(\mathcal{G},v) = \{v\}$. This ensures that we do not have attachment cycles. In other words, after a vertex chooses an attachment point from $T(\mathcal{G},v)$, v may be removed from $\mathcal{G}$ by replacing v with $r(v)$ in both E and T.

6.5.2 Defining resolution and embedded graphs of a general multiconnected graph $\mathcal{G}$

A (complete) resolution of a multiconnected graph $\mathcal{G}$ is *not* a graph. Instead, a (complete) resolution is a specific choice of attachment points for all wandering edges[7]. Thus a resolution r of $\mathcal{G}$ is a map $r : V(\mathcal{G}) \to \dot{V}(\mathcal{G})$ such that $r(v) \in T^+(\mathcal{G},v) \cap \dot{V}(\mathcal{G})$. Non-wandering vertices will always have $r(v) = v$. Two resolutions of $\mathcal{G}$, r_1 and r_2, are thus the same if $r_1(v) = r_2(v)$ for every $v \in V(\mathcal{G})$.

The number of distinct resolutions for $\mathcal{G}$ is

$$NR(\mathcal{G}) = \prod_{v\in V(\mathcal{G})} |T(\mathcal{G},v)|.$$

Definition 46. We further define $r(e) = (r(e_1), r(e_2))$.

Definition 47. Given a resolution r of the multiconnected graph $\mathcal{G}$, we may compute the (singly-connected) graph $r(\mathcal{G}) = (r \circ V(\mathcal{G}), r \circ E(\mathcal{G}))$. This graph is the embedded graph of $\mathcal{G}$ produced by the resolution r[8].

[7] I'm sticking with the idea that it is edges that wander, so that only degree 1 vertices can have equations. This causes problems, though, if wandering vertices can wander over other wandering vertices. We avoid this when enumerating embedded graphs by only allowing wandering vertices to choose non-wandering vertices as attachment points.

[8] Note that the other definition (of resolving *edges* instead of vertices) becomes problematic if the multiset $E(\mathcal{G})$ ever has edges with count>1, since it does not allow identical edges to be resolved differently.

6.5.3 Splits and connectedness in a multiconnected graph $\mathcal{G}$

For an edge $e = (u, v)$ in a singly connected graph G, we define $C(v, G)$ to be the set of vertices in g that are connected to v in G:

$$C(v, G) \quad = \quad \{w \in g.V | w \sim v \text{ in } g\}\,.$$

We also define the set $D(v, G) = C(v, G)^C$ to the be set of vertices of G that are *not* connected to v.

We extend these definitions to multiconnected graphs $\mathcal{G}$ by defining C (resp. D) to refer to those points that are connected (resp. disconnected) in every resolution of $\mathcal{G}$:

$$\begin{aligned} C(v, \mathcal{G}) &= \cap_r C(r(v), r(\mathcal{G})) \\ D(v, \mathcal{G}) &= \cap_r D(r(v), r(\mathcal{G})) \end{aligned}$$

The split induced by removing e in $\mathcal{G}$ can thus be defined in a single way, regardless of whether $\mathcal{G}$ is multiconnected or singly connected. We define the split induced by e in $\mathcal{G}$ to be $\sigma(e^t, \mathcal{G})|\sigma(e, \mathcal{G})$.

Definition 48. Here the set $\sigma(e, \mathcal{G})$ is defined as:

$$\sigma(e, \mathcal{G}) \quad = \quad C(e_2, \mathcal{G}/e) \cap D(e_1, \mathcal{G}/e)$$

6.5.4 Edge contraction of multiconnected graphs

First, we first define the replacement function $\cdot|_{X \to y}$ on vertices as:

$$s|_{X \to y} \quad = \quad \begin{cases} y & \text{if } s \in X \\ s & \text{otherwise.} \end{cases}$$

Then it follows that:

$$\begin{aligned} S|_{X \to y} &= \bigcup_{s \in S} s|_{X \to y} \\ &= \begin{cases} S & \text{if } S \cap X = \emptyset \\ \left(S \cap X^C\right) \cup \{y\} & \text{otherwise.} \end{cases} \end{aligned}$$

Second, given an edge e and a vertex v, we define a more specialized replacement function $|_{e \to v}$ on vertices, edges, and target sets. For a point w, we define $w|_{e \to v} = w|_{\{e_1, e_2\} \to v}$. For an edge f we define $f|_{e \to v}$ as $\{\emptyset\}$ if $e = f$, and $(f_1|_{e \to v}, f_2|_{e \to v})$ otherwise. Finally, for target sets T we define $T|_{e \to v}$ such that $T|_{e \to v}(e_1)$ and $T|_{e \to v}(e_2)$ are undefined, $T|_{e \to v}(v) = v$, and $T|_{e \to v}(w) = T(w)|_{e \to v}$ if $w \notin \{e_1, e_2, v\}$.

In theory, this definition would mean that, when contracting an edge e in which both endpoints wander to a vertex v, all targets of e_1 and e_2 are lost instead of being (perhaps) merged to create a larger target set for v. However, in this paper we only consider contracting doubly-non-wandering edges. In this case $T|_{e \to v}(v) = v$ is exactly what we want.

Third, given an edge e that is doubly non-wandering in $\mathcal{G}$, we define the contraction $\mathcal{G}|_{e \to v}$ of the graph $G = (E, V, T)$ as

$$\mathcal{G}|_{e \to v} \quad = \quad (V|_{e \to v}, (E/e)\,|_{e \to v}, T|_{e \to v}).$$

Here,

$$\begin{aligned} V|_{e,v} &= \bigcup_{w\in V} w|_{\{e_1,e_2\}\to v} \\ (E/e)|_{e,v} &= \bigcup_{f\in E/e} f|_{e\to v} \end{aligned}$$

Now, if $E(\mathcal{G})$ does not contain the edge e, then the only change here is to replace e_1 and e_2 with v. If e_1 and e_2 were originally in different connected components of $\mathcal{G}$, then this will connect the two originally separate components.

6.5.5 Contractions and resolutions together

Resolutions of $\mathcal{G}$ and resolutions of $\mathcal{G}|_{e\to v}$ have a particular relationship. First, given that e_1 and e_2 are non-wandering vertices of $\mathcal{G}$ and that v is not a vertex of G, then a resolution r of G naturally forms a resolution $r|_{e\to v}$ on $\mathcal{G}|_{e\to v}$.

Definition 49. If r is a resolution of a multiconnected graph $\mathcal{G}$, and e_1 and $\{e_1, e_2\}$ are non-wandering in $\mathcal{G}$ and $v \notin V(\mathcal{G})$, then we define the contracted resolution $r|_{e\to v}$ as follows:

- $r|_{e\to v}(e_1)$ and $r|_{e\to v}(e_2)$ are undefined.
- $r|_{e\to v}(v) = v$.
- $r|_{e\to v}(w) = r(w)|_{e\to v}$ otherwise.

Lemma 50. *Let $\mathcal{G}$ be a multiconnected graph, and let e by a doubly-nonwandering edge in $\mathcal{G}$, and let $v \notin V(\mathcal{G})$. Let s be a resolution of $\mathcal{G}|_{e\to v}$. Then there exists some resolution r of $\mathcal{G}$ such that $s = r|_{e\to v}$.*

Proof. We may construct an $r(w)$ such that:

$$r(w) = \begin{cases} \text{undefined} & \text{if } w = v \\ w & \text{if } w = e_1 \text{ or } w = e_2 \\ \begin{cases} e_1 \text{ or } e_2 & \text{if } s(w) = v \\ s(w) & \text{otherwise} \end{cases} & \text{otherwise.} \end{cases}$$

Thus thus number of such r's is 2^n, where n is the number of vertices w in $V(\mathcal{G}|_{e\to v})/v$ where $s(w) = v$.

Let us now check that $r|_{e\to v}(w) = s(w)$ for $w \in V(\mathcal{G})$.

1. $s(e_1)$ and $s(e_2)$ are undefined, since e_1 and e_2 are not in $V(\mathcal{G}|_{e\to v})$; $r|_{e\to v}(e_1)$ and $r|_{e\to v}(e_2)$ are undefined by the definition of $r|_{e\to v}$.

2. $s(v) = v$, since v is non-wandering; $r|_{e\to v}(v) = v$ by the definition of $r|_{e\to v}$.

3. If $w \notin \{e_1, e_2, v\}$ then $r|_{e\to v}(w) = r(w)|_{e\to v}$. Now, if $s(w) = v$, then $r|_{e\to v}(w) = e_1|_{e\to v}$ or $e_2|_{e\to v} = v$. Therefore, $r|_{e\to v}(w) = s(w)$. If $s(w) \neq v$, then $r|_{e\to v}(w) = r(w)|_{e\to v} = s(w)|_{e\to v}$. However, $s(w)|_{e\to v} = s(w)$, since $s(w) \in \dot{V}(\mathcal{G}|_{e\to v})$ and $\dot{V}(\mathcal{G}|_{e\to v})$ does not contain e_1 or e_2.

Therefore, for all $w \in V(\mathcal{G}|_{e\to v})$, $r|_{e\to v}(w) = s(w)$ and so $r|_{e\to v} = s$. [Also mention their domains].

$\square$

Theorem 51. *If $\{e_1, e_2\}$ are non-wandering in $\mathcal{G}$ and $v \notin V(\mathcal{G})$ and r is a resolution of $\mathcal{G}$, then*

$$r(w)|_{e\to v} \quad = \quad r|_{e\to v}(w|_{e\to v})$$

for any w in $V(\mathcal{G})$. Further, $r(w)|_{e\to v} = v$ if $w \in \{e_1, e_2\}$ or $r(w) \in \{e_1, e_2\}$. Otherwise, $r(w)|_{e\to v} = r(w)$.

Proof. There are three cases:

- **[w is on the edge]** If $w \in \{e_1, e_2\}$ then $r(w) = w$ because e_1 and e_2 are non-wandering, and so $r(w)|_{e\to v} = v$. Also $w|_{e\to v} = v$ and so $r|_{e\to v}(w|_{e\to v}) = r|_{e\to v}(v) = v$. Therefore, $r(w)|_{e\to v} = r|_{e\to v}(w|_{e\to v}) = v$ in this case.
- **[w resolves to the edge]** If $w \notin \{e_1, e_2\}$ but $r(w) \in \{e_1, e_2\}$ then $w|_{e\to v} = w$ and $r(w)|_{e\to v} = v$ and $r_{e\to v}(w|_{e\to v}) = r|_{e\to v}(w) = v$. Therefore, $r(w)|_{e\to v} = r|_{e\to v}(w|_{e\to v}) = v$ in this case.
- **[w is neither on the edge, nor resolves to it]** If $w \notin \{e_1, e_2\}$ and $r(w) \notin \{e_1, e_2\}$ then $w|_{e\to v} = w$ and $r(w)|_{e\to v} = r(w)$. Thus $r|_{e\to v}(w|_{e\to v}) = r|_{e\to v}(w) = r(w)$. Therefore, $r(w)|_{e\to v} = r|_{e\to v}(w|_{e\to v}) = r(w)$ in this case.

□

Corollary 52. *If $\{e_1, e_2\}$ are non-wandering in $\mathcal{G}$ and $v \notin V(\mathcal{G})$ and r is a resolution of $\mathcal{G}$, and $W \in V(G)$ then*

$$r(W)|_{e\to v} \quad = \quad r|_{e\to v}\left(W|_{e\to v}\right).$$

Theorem 53. *If $\{e_1, e_2\}$ are non-wandering in $\mathcal{G}$ and $v \notin V(\mathcal{G})$ and $e \in E(\mathcal{G})$ and r is a resolution of $\mathcal{G}$ then $r(\mathcal{G})|_{e\to v} = r|_{e\to v}(\mathcal{G}|_{e\to v})$.*

Using the vertex-based resolution scheme, we have:

$$\begin{aligned} r(\mathcal{G})|_{e\to v} &= \left(r\left(V(\mathcal{G})\right), r\left(E(\mathcal{G})\right)\right)|_{e\to v} \\ &= \left(r\left(V(\mathcal{G})\right)|_{e\to v}, \left[r\left(E(\mathcal{G})\right)/e\right]|_{e\to v}\right) \\ &= \left(r\left(V(\mathcal{G})\right)|_{e\to v}, r\left(E(\mathcal{G})/e\right)|_{e\to v}\right) \\ &= \left(r|_{e\to v}\left(V(\mathcal{G})|_{e\to v}\right), r|_{e\to v}\left(\left[E(\mathcal{G})/e\right]|_{e\to v}\right)\right). \end{aligned}$$

We also have

$$\begin{aligned} r|_{e\to v}(\mathcal{G}|_{e\to v}) &= r|_{e\to v}(V(\mathcal{G})|_{e\to v}, \left[E(\mathcal{G})/e\right]|_{e\to v}, T(\mathcal{G})|_{e\to v}) \\ &= \left(r|_{e\to v}\left(V(\mathcal{G})|_{e\to v}\right), r|_{e\to v}\left[E(\mathcal{G})/e\right]|_{e\to v}\right). \end{aligned}$$

Therefore $r(\mathcal{G})|_{e\to v} = r|_{e\to v}(\mathcal{G}|_{e\to v})$.

Note: Resolutions on G can be divided into equivalence classes based on whether a contraction $\cdot|_{e\to v}$ yields the same projected resolution.

Lemma 54. *If r is a resolution of $\mathcal{G}$ then $r(\mathcal{G}/e) = r(\mathcal{G})/r(e)$*

Proof. So, by the definition of r we have:

$$\begin{aligned} r(\mathcal{G}/e) &= (r(V(\mathcal{G}/e)), r(E(\mathcal{G}/e))) \\ &= (r(V(\mathcal{G})), r(E(\mathcal{G})/e)). \end{aligned}$$

Now, $E(\mathcal{G})$ is a multiset. Therefore, if 1 or more edges in $E(\mathcal{G})$ map to $r(e)$, then $r(e)$ will have a count of 1 less in $r(E(\mathcal{G})/e)$ than in $r(E(\mathcal{G}))$. If $E(\mathcal{G})$ contains no edges that map to $r(e)$, then $r(E(\mathcal{G})/e) = r(E(\mathcal{G})) = r(E(\mathcal{G}))/r(e)$. In both cases, $r(E(\mathcal{G})/e) = r(E(\mathcal{G}))/r(e)$. □

6.5.6 Connectedness in multiconnected graphs

Single connected graphs For singly connected graphs, we have

- for any $x, y \in V(G)$: $x \sim y$ in $G \implies x|_{e\to v} \sim y|_{e\to v}$ in $G|_{e\to v}$
- for any $x, y \in V(G)$: $x \not\sim y$ in $G \implies x|_{e\to v} \not\sim y|_{e\to v}$ in $G|_{e\to v}$ if $e \in E(G)$.
- Therefore, we also have that

  1. for any $x, y \in V(G)$ if $e \in E(G)$ then $x|_{e\to v} \sim y|_{e\to v}$ in $G|_{e\to v} \implies x \sim y$ in G.
  2. for any $x, y \in V(G)$ if $e \in E(G)$ then ($x \sim y$ in $G \iff x|_{e\to v} \sim y|_{e\to v}$ in $G|_{e\to v}$)

Resolutions of multiconnected graphs We show above that $r(\mathcal{G}/e) = r(\mathcal{G})/r(e)$. Let us now assume that the contracted edge e is doubly non-wandering in a multiconnected graph $\mathcal{G}$, so that $r(e) = e$. Then $r(\mathcal{G}/e) = r(\mathcal{G})/e$.

Now, let us consider the connectivity of $r(x)$ and $r(y)$ in the singly connected graph $r(\mathcal{G})$:

$$r(x) \sim r(y) \text{ in } r(\mathcal{G}) \iff r(x)|_{e\to v} \sim r(y)|_{e\to v} \text{ in } r(\mathcal{G})|_{e\to v}.$$

if $e \in E(r(\mathcal{G}))$ and $r(x), r(y) \in V(r(\mathcal{G}))$. Now, $E(r(G)) = r(E(G))$ and $V(r(\mathcal{G})) = r(V(\mathcal{G}))$. Thus the condition is that $e \in r(E(\mathcal{G}))$ and that $r(x), r(y) \in r(V(\mathcal{G}))$. A sufficient condition is thus that $x, y \in V(\mathcal{G})$ and $e \in E(\mathcal{G})$.

Since $r(\cdot)|_{e\to v} = r|_{e\to v}(\cdot|_{e\to v})$ we also have for any $x, y \in V(\mathcal{G})$ and doubly non-wandering edge $e \in E(\mathcal{G})$:

$$r(x) \sim r(y) \text{ in } r(\mathcal{G}) \iff r|_{e\to v}(x|_{e\to v}) \sim r|_{e\to v}(y|_{e\to v}) \text{ in } r|_{e\to v}(\mathcal{G}|_{e\to v}).$$

Resolutions of multiconnected graphs with a split We now consider connectivity in $r(\mathcal{G})$ and $r(\mathcal{G})|_{e\to v}$ when an edge f in a multiconnected graph G is cut. We thus replace $\mathcal{G}$ with $\mathcal{G}/f$, obtaining the new conditions: $x, y \in V(\mathcal{G}/f) = V(\mathcal{G})$ and $e \in E(\mathcal{G}/f)$. This is the same as the condition that $x, y, \in V(\mathcal{G})$ and $e \in \mathcal{G}$ and $f \neq e$. Under these conditions, we have:

$$r(x) \sim r(y) \text{ in } r(\mathcal{G}/f) \iff r|_{e\to v}(x|_{e\to v}) \sim r|_{e\to v}(y|_{e\to v}) \text{ in } r|_{e\to v}([\mathcal{G}/f]\,|_{e\to v}).$$

Since $(\mathcal{G}/f)|_{e\to v} = \mathcal{G}|_{e\to v}/f|_{e\to v}$ we finally have:

$$r(x) \sim r(y) \text{ in } r(\mathcal{G}/f) \iff r|_{e\to v}(x|_{e\to v}) \sim r|_{e\to v}(y|_{e\to v}) \text{ in } r|_{e\to v}(\mathcal{G}|_{e\to v}/f|_{e\to v}).$$

Thus we could write:

$$r(x) \sim r(y) \text{ in } r(\mathcal{G}/f) \iff r'(x') \sim r'(y') \text{ in } r'(\mathcal{G}'/f')$$

where $(\cdot)'$ indicates $(\cdot)\,|_{e\to v}$. However, note that the requirement that $f \neq e$ is crucial, since $\mathcal{G}'/f'$ is not split into two components by f' if $e = f$.

Mapping between resolutions of $\mathcal{G}$ and resolutions of $\mathcal{G}|_{e\to v}$ We now seek to determine what $\sigma(f|_{e\to v}, \mathcal{G}|_{e\to v})$ tells us about $\sigma(f, \mathcal{G})$.

Lemma 55. *If $\mathcal{G}$ is a multiconnected graph, $x \in V(\mathcal{G})$, $v \notin V(\mathcal{G})$ and $e \in E(\mathcal{G})$, then $C(x, \mathcal{G})|_{e\to v} = C(x|_{e\to v}, \mathcal{G})$ and $D(x, \mathcal{G})|_{e\to v} = D(x|_{e\to v}, \mathcal{G}|_{e\to v})$.*

Proof. We begin by considering $y \in C(x, \mathcal{G})$, and note that for any resolution r' on $\mathcal{G}|_{e\to v}$, there exists a resolution r on $\mathcal{G}$ such that $r|_{e\to v} = r'$. Now, $r(x) \sim r(y)$ in $r(\mathcal{G})$ by our premise, and so $r'(x|_{e\to v}) \sim r'(y|_{e\to v})$ in $r'(\mathcal{G}|_{e\to v})$. Therefore $y|_{e\to v} \in C(x|_{e\to v}, \mathcal{G}|_{e\to v})$ and $C(x, \mathcal{G})|_{e\to v} \subseteq C(x|_{e\to v}, \mathcal{G}|_{e\to v})$.

Likewise, if $y \notin C(x, \mathcal{G})$ then there is some resolution r such that $r(x) \not\sim r(y)$ in $r(\mathcal{G})$. Therefore, there is some resolution $r|_{e\to v}$ such that $r|_{e\to v}(x|_{e\to v}) \not\sim r|_{e\to v}(y|_{e\to v})$ in $r(\mathcal{G}|_{e\to v})$, and so $y|_{e\to v} \notin C(x|_{e\to v}, \mathcal{G}|_{e\to v})$.

Now, if $y' \in C(x|_{e\to v}, \mathcal{G}|_{e\to v})$, then there must exist a y in $V(\mathcal{G})$ such that $y|_{e\to v} = y'$. We must have $y \in C(x, \mathcal{G})$, since the opposite would contradict our premise that $y' \in C(x|_{e\to v}, \mathcal{G}|_{e\to v})$. Therefore, $y' \in C(x, \mathcal{G})|_{e\to v}$, and $C(x, \mathcal{G})|_{e\to v} \subseteq C(x|_{e\to v}, \mathcal{G}|_{e\to v})$.

Combining the two $\subseteq$ statements, we get $C(x, \mathcal{G})|_{e\to v} = C(x|_{e\to v}, \mathcal{G}|_{e\to v})$. By symmetry, we also have $D(x, \mathcal{G})|_{e\to v} = D(x|_{e\to v}, \mathcal{G}|_{e\to v})$. $\square$

Lemma 56. *If $\mathcal{G}$ is a multiconnected graph, and $e, f \in E(\mathcal{G})$ and $e \neq f$ and e is doubly non-wandering in $\mathcal{G}$, then*

$$\sigma(f, G) \quad = \quad \begin{cases} \sigma(f|_{e\to v}, G|_{e\to v}) \cup \{e_1, e_2\}/v & \text{if } v \in \sigma(f|_{e\to v}, G|_{e\to v}) \\ \sigma(f|_{e\to v}, G|_{e\to v}) & \text{otherwise.} \end{cases}$$

Proof. Now, we note that $\sigma(e, \mathcal{G}) = C(e_2, \mathcal{G}/e) \cap D(e_1, \mathcal{G}/e)$. Therefore

$$\begin{aligned} \sigma(f, \mathcal{G})|_{e\to v} &= [C(f_2, \mathcal{G}/f) \cap D(f_1, \mathcal{G}/f)] \\ &= C(f_2, \mathcal{G}/f)|_{e\to v} \cap D(f_1, \mathcal{G}/f)|_{e\to v}. \end{aligned}$$

Now, if $f \in \mathcal{G}/e$ and e is doubly non-wandering in $\mathcal{G}$, then we have

$$\begin{aligned} \sigma(f, \mathcal{G})|_{e\to v} &= C(f_2|_{e\to v}, (\mathcal{G}/f)\,|_{e\to v}) \cap D(f_1|_{e\to v}, (\mathcal{G}/f)\,|_{e\to v}) \\ &= C(f_2|_{e\to v}, \mathcal{G}|_{e\to v}/f|_{e\to v}) \cap D(f_1|_{e\to v}, \mathcal{G}|_{e\to v}/f|_{e\to v}) \\ &= \sigma(f|_{e\to v}, \mathcal{G}|_{e\to v}). \end{aligned}$$

This tells us how to construct $\sigma(f|_{e\to v}, \mathcal{G}|_{e\to v})$ from $\sigma(f, \mathcal{G})$. However, how do we construct $\sigma(f, \mathcal{G})$ from $\sigma(f|_{e\to v}, \mathcal{G}|_{e\to v})$? If $\sigma(f|_{e\to v}, \mathcal{G}|_{e\to v})$ does not contain v, then $\sigma(f, \mathcal{G}) = \sigma(f|_{e\to v}, \mathcal{G}|_{e\to v})$. If $\sigma(f|_{e\to v}, \mathcal{G}|_{e\to v})$ does not contain v, then $\sigma(f, \mathcal{G})$ will be $\sigma(f|_{e\to v}, \mathcal{G}|_{e\to v})/v$ plus either e_1 or e_2 or both. Now, since $e \neq f$ then e_1 and e_2 are always connected to each other in $\mathcal{G}/f$. Therefore, they will either both be in $C(f_2, \mathcal{G}/f)$ and $D(f_1, \mathcal{G}/f)$, or both be absent. Therefore, they will either both be present in $\sigma(f, \mathcal{G})$ or both absent. $\square$

6.5.7 Proof of split representation in $G(\Sigma + A)$

The proof has two parts. First, for edges that are not A, each split in Σ should be essentially unchanged in $\Sigma + A$. However, for the split on the edge A, we must do induction, as follows:

Definition 57. We define the statement $SR(\Sigma)$ to be

$$\begin{aligned} SR(\Sigma) \quad = \quad & \nu(C) \in \sigma(e(B), G(\Sigma)) \text{ for any } B, C \in \check{\Sigma} \text{ where } B < C \text{ or } B \downarrow C \\ & \nu(C) \notin \sigma(e(B), G(\Sigma)) \text{ for any } B, C \in \check{\Sigma} \text{ where } C \downarrow B \text{ or } C \perp B. \end{aligned}$$

Claim 58. We then claim that if Σ is a multiconnected split set, and $\Sigma + A$ is a multiconnected split set where A is doubly non-wandering in $\Sigma + A$, then $SR(\Sigma) \implies SR(\Sigma + A)$.

Note $G(\Sigma) = G(\Sigma + A)|_{e(A) \to J(A)}$

Reasoning The reasoning behing this argument goes as follows.

If $A < C$ or $A \downarrow C$ then either $A < M \leq C$ or $A < M \downarrow C$ for some $M \in \min_{<_o} S(A, \Sigma)$. By the previous iteration of the theorem, $\nu(M) \sim \nu(C)$ by any resolution r when $\nu(M)$ is cut. *Why can we assume this means that $\nu(A) \sim \nu(C)$ when $e(A)$ is cut?*

Case 1: If $A \ll M$ then $\nu(A) \sim \nu(M)$ in $G(\Sigma)/e(A)$ and therefore in any resolution.

Case 2: If $M^t \notin NW(\Sigma + A)$ then the attachment points of M^t either equal $\nu(A, \Sigma + A)$ or are behind M_i' some that $A \ll M_i'$. **...here...**

6.6 Rest of proof of representation

Lemma. *If Σ is a multiconnected split set and $A \in \check{\Sigma}$ and neither A nor A^t directly wander in Σ, then it is never the case that $\nu(A^t) = \nu(A)$.*

Proof. If $\nu(A^t) = \nu(A)$ then $(A, A^t) \in N(\Sigma)$. Since $A \neq A^t$, this would require that $A \ll A^t$. However, $A < A^t$ is always false, so this cannot happen. $\square$

Theorem 59. *If Σ is a multiconnected split set and (S_1, S_2) is a multi-edge of $G(\Sigma)$, then S_1 and S_2 are disjoint.*

Proof. The edge (S_1, S_2) must equal $(T(A^t), T(A))$ for some split $A \in \check{\Sigma}$ by the definition of $E(\Sigma)$. If S_1 and S_2 are not disjoint, then there must be

Case 1. $A^t \in NW(\check{\Sigma})$ and $A \in NW(\check{\Sigma})$

Then $(S_1, S_2) = (\nu(A^t), \nu(A))$. If S_1 and S_2 overlap, then $\nu(A^t) = \nu(A)$, and by Lemma 6.6 this cannot happen. Therefore, S_1 and S_2 are disjoint in this case.

Case 2. $A^t \in NW(\check{\Sigma})$ and $A \notin NW(\check{\Sigma})$

Then $S_1 = \nu(A^t)$ and S_2 contains $\nu(A^t)$. This would require that there is some $B \in NW(\check{\Sigma})$ such that $A \Downarrow B$ and $(A^t, B) \in N(\Sigma)$. Now, $A^t \neq B$ because $A \downarrow B$. Furthermore, $A \not< B$ because $A \downarrow B$. Therefore $S_1$and S_2 are disjoint in this case.

Case 3. $A^t \notin NW(\check{\Sigma})$ and $A \in NW(\check{\Sigma})$.

By symmetry, the previous case proves this case.

Case 4. $A^t \notin NW(\check{\Sigma})$ and $A \notin NW(\check{\Sigma})$

Then there must exist splits $C \in NW(\check{\Sigma})$ and $D \in NW(\check{\Sigma})$ and $A^t \Downarrow C$ and $A \Downarrow D$ such that $\nu(C) = \nu(D)$ and thus $(C, D) \in N(\Sigma)$. Since $C \perp D$, we cannot have either $C = D$ or $C < D$, and thus $(C, D) \in N(\Sigma)$ cannot occur. Therefore, S_1 and S_2 are disjoint in this case

Since $T(A^t)$ and $T(A)$ must be disjoint in all cases, S_1 and S_2 must be disjoint. $\square$

Lemma 60. *If Σ is a multiconnected split set and $A \in \check{I}(\Sigma)$, then $\Sigma - A$ fails to be a multiconnected split set if and only if A and A^t both directly wander in Σ.*

Proof. For any such edge A, let us assume that $\Sigma - A$ is not a multiconnected split set. All the pairs of splits in $\Sigma - A$ must satisfy the pairwise constraints, since they did so in Σ. Furthermore, A is not a leaf split, and so removing A will not violate the requirement that all leaf splits must be in Σ. Therefore, $\Sigma - A$ must contain some pairs of splits B and C such that

1. $B \perp C$
2. $A^t \downarrow B$ and $A \downarrow C$
3. There is no split $D \in \check{\Sigma} - A$ that $D^t \downarrow B$ and $D \downarrow C$.

Now, if A^t does not wander directly over B in Σ, then there is some split $E \in \Sigma$ such that $A^t < E \downarrow B$. Furthermore, we also have $E^t < A \downarrow C$. The split E would also be in $\Sigma - A$ since $E \neq A$. This would contradict point 3 above, and so $A^t \Downarrow B$ in Σ. The same argument can be repeated to demonstrate that $A \Downarrow C$ in Σ. Thus, if Σ is multiconnected split set but $\Sigma - A$ is not, then A and A^t must both directly wander in Σ.

Now, assume that A and A^t directly wander in Σ. There are therefore some splits B and C in Σ such that $A^t \Downarrow B$ and $A \Downarrow C$. Since $B_1 \cup B_2 \subseteq A_1$ and $C_1 \cup C_2 \subseteq A_2$, we have $B \perp C$. Therefore, if $\Sigma - A$ is a multiconnected split set, there must be some split $D \in \Sigma - A$ such that $D^t \downarrow B$ and $D \downarrow C$. But then, by Lemma 78, either $D < A$ or $A < D$. If $D < A$, then $A^t < D^t \downarrow B$ and A^t does not directly wander. If $A < D$, then $A < D \downarrow C$, and A does not directly wander. Since both cases contradict our assumption that A and A^t directly wander in Σ, there can be no such split $D \in \Sigma - A$, and $\Sigma - A$ fails to be a multiconnected split set. Therefore, if A and A^t directly wander in Σ, then $\Sigma - A$ is not a multiconnected split set. □

Definition 61. A contraction of a simple edge $e = (v_1, v_2)$ in a multiconnected graph G that contains e is a new graph G' obtained by

1. Removing e from the edges of G
2. Removing the vertices v_1 and v_2 and replacing them with a (possibly new) vertex w
3. Replacing occurrences of v_1 and v_2 with w in all multi-edges.

Note that, if an edge (S_1, S_2) has $S_1 = \{v_1, v_2\}$ then this process results in $S'_1 = \{w\}$ since S'_1 is a set, not a multi-set. Therefore, contraction can make complex[9] edges into simple[10] edges.

Lemma. *If Σ is a multiconnected split set containing A and B, and $\nu(A, \Sigma) \neq \nu(B, \Sigma)$ then either*

- $A \not< B^t$
- *$A < B^t$ and there exists a $C \in \check{\Sigma}$ such that $A < C < B^t$.*
- *A directly wanders in Σ*
- *B directly wanders in Σ*

Proof. This follows directly from the definition of the equivalence relation $N(\Sigma)$ and from the definition of $A \ll B^t$ in Σ. □

[9]Use this as a definition!

[10]Use this as a definition!

Lemma 62. *If Σ and $\Sigma + A$ are multiconnected split sets and $A \in I(\Sigma + A)$ and $B \in \check{\Sigma}$, then $\nu(B, \Sigma + A) \subseteq \nu(B, \Sigma) \cup \{A, A^t\}$.*

Proof. Suppose that $\nu(B, \Sigma + A) \not\subseteq \nu(B, \Sigma) \cup \{A, A^t\}$. Then there exists some ordered split $C \in \nu(B, \Sigma + A)$ such that $C \notin \nu(B, \Sigma) \cup \{A, A^t\}$. Therefore $C \notin \{A, A^t\}$ and so $C \in \check{\Sigma}$. Furthermore, $C \in NW(\Sigma + A)$ does not wander directly over any split in $\Sigma + A$, and so does not wander over any split in Σ, and so $C \in NW(\check{\Sigma} + A + A^t)$. Since $C \in \nu(B, \Sigma + A)$, there exists no ordered split $E \in \check{\Sigma}$ such that $B < E < C^t$. Removing the split A from Σ cannot introduce a new split between B and C^t and so we must have $B \lll C^t$ in Σ, and thus $C \in \nu(B, \Sigma)$. This contradicts our assumption that $C \notin \nu(B, \Sigma) \cup \{A, A^t\}$. Therefore we conclude by contradiction that $\nu(B, \Sigma + A) \subseteq \nu(B, \Sigma) \cup \{A, A^t\}$. □

Claim 63. Also, the size of $\nu(B, \Sigma)$ cannot increase by adding an edge A since, even if A or A^t becomes part of $\nu(B, \Sigma + A)$, at least one split must now be behind A in $\Sigma + A$, and thus that split must leave $\nu(B, \Sigma + A)$.

Lemma 64. *If Σ is a multiconnected split set, and $A \in I(\Sigma \check{+} A)$ and neither A nor A^t directly wander in $\Sigma + A$, then any node in $\Sigma + A$ that is not $\nu(A)$ or $\nu(A^t)$ is unchanged.*

Proof. Suppose that some node $v \in V(\Sigma + A)$ is not present in $V(\Sigma)$. Since v is an equivalence class of ordered splits that do not wander in Σ, then there exists some $C \in NW(\Sigma \check{+} A)$ such that $v = \nu(C, \Sigma + A)$. Since this node is not present in $V(\Sigma)$ then we must have that $\nu(C, \Sigma + A) \neq \nu(C, \Sigma)$. Since $\nu(C, \Sigma + A)$ does not contain either A or A^t, then by Lemma 62 we must have $\nu(C, \Sigma + A) \subseteq \nu(C, \Sigma)$.

Therefore there must be some D in $\nu(C, \Sigma)$ that is not in $\nu(C, \Sigma + A)$. Since $D \in \Sigma \check{+} A$, then either $D \notin NW(\Sigma \check{+} A)$ or $C \not\lll D^t$ in $\Sigma + A$.

Case 1. Suppose $C \not\lll D^t$ in $\Sigma + A$. Since $C \lll D^t$ in Σ, then we must have some E such that $C < E < D^t$. However, since E is in $\Sigma \check{+} A$ but not in $\check{\Sigma}$, then E must be either A or A^t. Furthermore, since we cannot have both $C < A$ and $C < A^t$, we must have either $C \lll A \lll D^t$ or $C \lll A^t \lll D^t$. However, these cases would imply that $\nu(C, \Sigma + A)$ equals either $\nu(A^t, \Sigma + A)$ or $\nu(A, \Sigma + A)$, both of which contradict the asumptions of this lemma. Therefore, this case cannot occur.

Case 2. Suppose $D \notin NW(\Sigma \check{+} A)$. Then $D \Downarrow A$ or $D \Downarrow A^t$. Since each implies the other, we have $D \Downarrow A$ and $D \Downarrow A^t$, but D does not wander over any other split in $\Sigma \check{+} A$. Then there is no $E \in \Sigma \check{+} A$ such that $C < E < D^t$ because this is true in Σ, and adding A cannot change this since $D \downarrow A$ and $D \downarrow A^t$.

□

Definition 65. We define the set of edges to the left of A as $L(A, \Sigma) \equiv \{B \in \Sigma : B < A \vee A^t \downarrow B\}$.

Lemma 66. *If B is a minimal size split in $L(A, \Sigma) \cap I(\Sigma)$, then B^t does not wander over any split in Σ, and B does not directly wander in Σ.*

Proof. Consider any B that is a mininimal size split in $L(A, \Sigma) \cap I(\Sigma)$.

Case 1. Assume that there is some split $C \in \check{\Sigma}$ such that $B^t \downarrow C$. Now if $A^t \downarrow B$, then $A^t \downarrow C$ since $B^t \downarrow C$. Alternatively, if $B < A$, then $A^t \downarrow C$, since $B \downarrow C$. Thus, in either case, $C \in L(A, \Sigma)$. Thus, in either case $C \in L(A, \Sigma)$. Furthermore, since $B^t \downarrow C$, C cannot be a full split. C must

be an informative split, since it would otherwise be implied by one of the leaf splits[11]. Therefore, $C \in L(A,\Sigma) \cap I(\Sigma)$.
Now, since $B \downarrow C$, the size $|C_1 \cup C_2|$ of C must be smaller than the size of B. This contradicts the assumption that no split in $L(A,\Sigma) \cap I(\Sigma)$ has a smaller size than B. Therefore B^t does not wander over any split in Σ.

Case 2. Assume that $B \Downarrow C$. Then $B \downarrow C$ and $B < A$. Then either,

- $C < A$. This cannot occur, because $|C| < |B|$ and $C \in L(A,\Sigma) \cap I(\Sigma)$
- $C^t < A$. This cannot occur for the same reason.
- $A \downarrow C$. This cannot occur, because it would mean $B < A \downarrow C$. This is *indirect* wandering.
- $A^t \downarrow C$. This cannot occur, because $|C| < |B|$ and $C \in L(A,\Sigma) \cap I(\Sigma)$
- $A \perp C$. In this case there must be some D such that $B < D \downarrow C$. This is *indirect* wandering.

Therefore, B does not directly wander in Σ.

□

Lemma 67. *If Σ is a multiconnected split set, then removing a minimal size split $B \in L(A,\Sigma) \cap I(\Sigma)$ from Σ yields a split set $\Sigma - B$ which is also a multiconnected split set.*

Proof. Lemma 66 implies that neither B or B^t directly wander in Σ. Therefore Lemma 60 implies that $\Sigma - B$ is a multiconnected split set since $B \in I(\Sigma)$. □

Lemma 68. *67For any edge $A \in \Sigma$, the multiconnected split set Σ can be transformed to a multiconnected split set $\Sigma' \equiv \Sigma \cap (L(A,\Sigma) \cap I(\Sigma))^C$ by incrementally removing splits that do not directly wander. Each split set in this sequence is also a multiconnected tree.*

Proof. Repeatedly removing a minimal size split of $L(A,\Sigma) \cap I(\Sigma)$ always results in a new multiconnected tree by Lemma 67. By Lemma 66, such splits do not directly wander in Σ.

Since this procedure can be continued as long as $|L(A,\Sigma_i) \cap I(\Sigma_i)| > 0$, it must eventually terminate as some Σ' where $|L(A,\Sigma') \cap I(\Sigma')| = 0$. Such a split set contains every split that in Σ that is not in $L(A,\Sigma) \cap I(\Sigma)$, but none that are, and therefore equals $\Sigma \cap (L(A,\Sigma) \cap I(\Sigma))^C$. □

Lemma 69. *The graph $G(\Sigma)$ contains no self-loops.*

Proof. Each edge connects vertices (A^t, A). For this edge to become a self-loop when edges are joined to form nodes, it would be necessary to identify A and A^t in step 4. However, this would require that $A \sim A^t$ according to the identity relation of $\{(A,B) : A = B \vee A \ll B^t\}$. However, $A \neq A^t$ and $A \not\ll A^t$, so $A \not\sim A^t$. Therefore, the graph $G(\Sigma)$ cannot contain a self-loop. □

Claim. If Σ is a multiconnected split set, and $\Sigma' \subset \Sigma$ is also a multiconnected split set, then there is a sequence of $\Sigma_1 = \Sigma, \Sigma_2, \ldots, \Sigma_n = \Sigma'$ such that each Σ_{i+1} removes a single edge A_i from Σ_i, where A_i does not directly wander in $G(\Sigma)$. Thus there is a sequence of graphs $G(\Sigma_i)$ from Σ to Σ' that is formed by contracting non-wandering edges.

Proof. <insert> □

Remark 70. Perhaps we should stipulate that, for multiconnected graphs, you can only have equations on degree-1 nodes.

[11] Improve this justification

Let us generalize the proof that a collection of pairwise compatible splits can be inserted into a single tree, and claim that

Claim 71. A collection of extended-compatible splits can be combined into a multiconnected tree graph. We extend the concept of extension so that we can extend a tree with partial splits, leading to some edges with multiple attachment points. These extensions correspond to embedded graphs that are trees, each of which is extended by an edge that implies the partial split.

Proof. OK, so take a split set Σ and its graph $G(\Sigma)$. Any new split A such that (i) $\Sigma + A$ is a multiconnected tree and (ii) A does not wander over any $B \in \Sigma$ has the property that A maps to some specific node in $G(\Sigma)$. We assume that embedded graphs of $G(\Sigma)$ are trees. Therefore, this edge maps to a node in each of these trees, and extends each of the tree by an edges that implies A. □

Claim 72. A contraction of $\mathcal{G}$ implies a contraction of any $g \in \mathcal{G}$.

Proof. Given that the edge e being contracted has two endpoints in g that have not been made identical by the process of resolving wandering edges, then, yes, contracting e in $\mathcal{G}$ must correspond to a contraction in g also.

Interestingly, I think that it would be useful to define a contraction so that we only "contract" edges where the endpoints are not equal. This way, contraction cannot (a) remove cycles (b) is always the reverse of some extension and (c) always reduces the number of nodes by 1. (Check Semple & Steel book). □

Claim 73. An extension of $\mathcal{G}$ implies an extension of any $g \in \mathcal{G}$.

Proof. Well, this becomes problematic, if you add an $A \downarrow B$ where $C < A$ and $D < A$. So, only add non-wandering edges. □

6.7 Graph-related Theorems

- Lemma 83.
- Lemma 89.
- Theorem **??**.

6.8 Splits implied by multiconnected trees

6.8.1 Edges in embedded graphs imply splits that are the same or more resolved

First note that for any edge a inducing a split A on a multiconnected graph G, the corresponding split A' induced by the corresponding edge a' in any embedded graph G' of G must imply A. This is because the edge $a' = (x', y')$ must have x' connected to all leaves in A_1 and y' connected to all leaves in A_2 when the edge a' is cut.

6.8.2 Counter-examples: split sets that cannot be represented

Consider a multiconnected graph G with edges a and b that induces splits A and B, respectively. Suppose that $A = 12|345$ and $B = 123|45X$. Now, this means that some embedded trees have $X12|345$ and some have $12|X345$, whereas all embedded trees have $123|45X$. But actually, $X12|345$ is not compatible with $123|45X$. Therefore, a multi-connected graph cannot have edges that are not close in this way.

To generalize this argument, if a split collection Σ contains a split A, and Σ jointly implies another split A' with $A' \implies A$ but $A' \neq A$, then the collection Σ cannot be represented by a multiconnected graph. For suppose (without loss of generality) that A' implies $x + A_1|A_2$. The fact that the graph contains an edge representing A means that some embedded trees display $x + A_1|A_2$ and some embedded trees display $A_1|A_2 + x$. But this contradicts the hypothesis that Σ jointly implies A'.

6.9 Questions

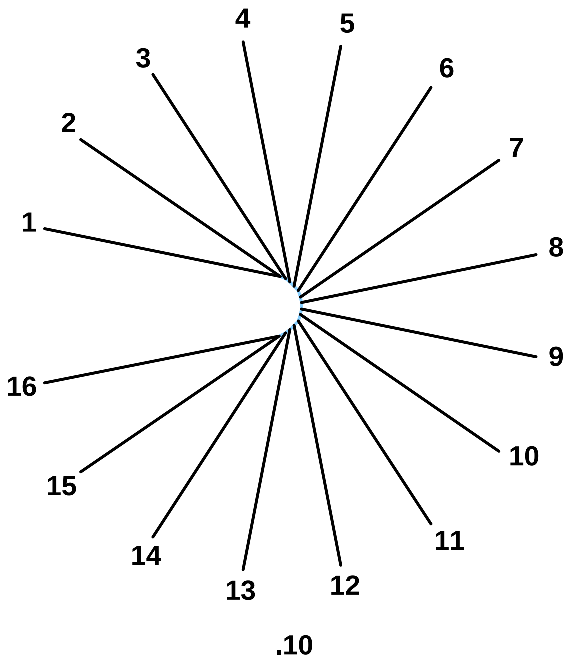
- Q1: How does resolving a multiconnected tree relate to the restrictions on resolving a multi-connected split set?
  - The only restriction is that we can't resolve $A \Downarrow B$ if $A \downarrow C \downarrow B$. So, what happens if we do this anyway?
    - Basically, we would be saying that $A' < B$ but $A \downarrow C$, and C can attach on both sides of B.
    - This could be resolved *either* by restricting C also (so that it is on the left side of B), *or* by making A stop wandering over C.
- Q2: How do I prove that *this* way of constructing a graph is what corresponds to the multiconnected split set? (Instead of just stipulating that this is the graph that we will be considering.)
- Q3: Can I just demonstrate that $G(\Sigma)$ represents Σ w/o using any of the multiconnected split set theorems?
  - Well, in order to work on the graph, we have to use the split relationships anyway: so, we may as well do things in terms of splits, anyway. (?)
  - Can I just demonstrate that $G(\Sigma)$ represents Σ w/o using the *rules* on the multiconnected split set?
  - Would trying to forge ahead via brute force lead to insight into why we need the rules?

7 Algorithm for finding supported splits

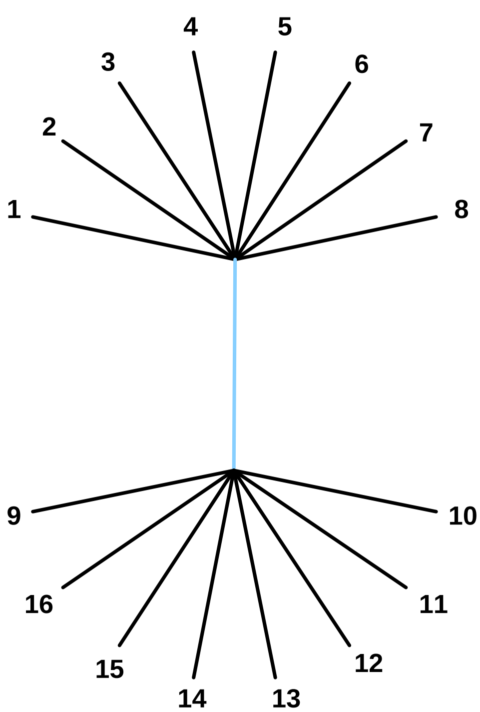
7.1 Estimating branch lengths for the partial splits?

8 Results

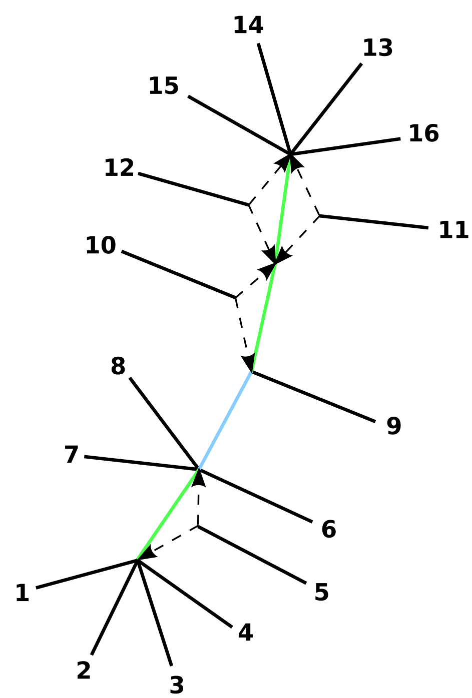
Here we show some examples of how the multiconnected tree reveals hidden structure in posterior distributions from real and simulated data sets.



(a) True tree

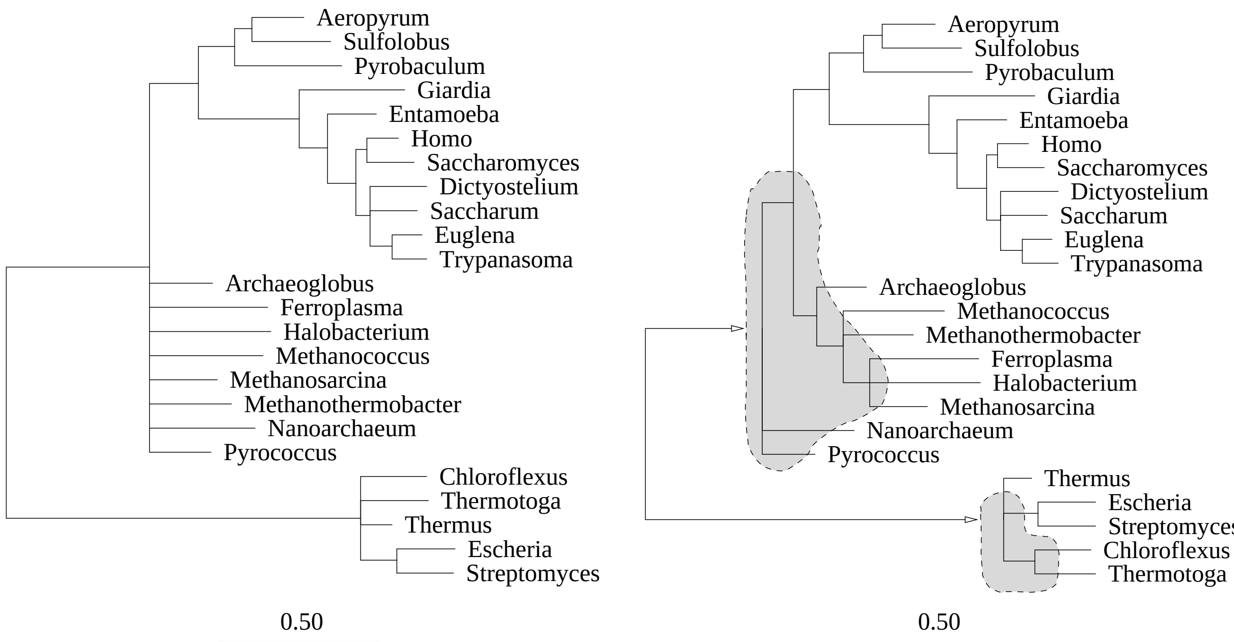


(b) 95% Majority Consensus



(c) 95% Extended Majority Consensus

Figure 13: Multiconnected trees help resolve structure in trees with short branches. Sequences were simulated on the true tree (a) that has very short internal branches. Using the 95% majority consensus (b) only one internal branch is discovered. However, the extended majority consensus (c) reveals three new branches by displaying partial splits. Partial splits can group adjacent full splits, thus yielding longer branches with sufficient support even the full splits correspond to very short branches.



(a) 95% majority consensus

(b) 95% Extended majority consensus

Figure 14: EF-Tu 24 taxon tree: Majority consensus and Extended Majority Consensus. Since the Eubacteria and the Archaea are separated by a long branch, the support for each group is strong, but each group provides low precision in estimating the root of the other group. (a) Because of this, the $M_{0.95}$ consensus tree is not able to represent much structure in the phylogeny of the Archaea, leading to a comb-like figure. (b) Because the $M'_{0.95}$ consensus tree is able to represent partial splits, it is able to reveal previously hidden structure in the posterior distribution. All branches inside each cloud correspond to partial splits, and the extent of the clouds give a rough idea of the range over which the wandering branch may attach.

8.1 Example 1 - Simulated data / Caterpillar tree

(See figure 13)

8.2 Example 2: EF-Tu / bad reciprocal rooting

(See figure 14)

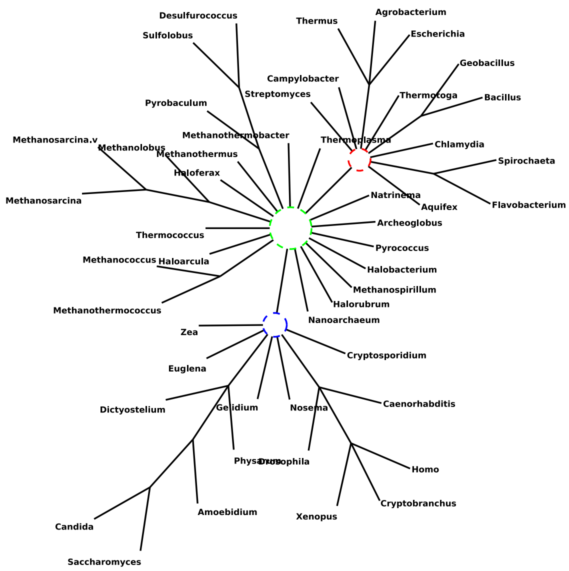
8.3 Example 4: 5s rRNA / 48 taxa

(See figure 15)

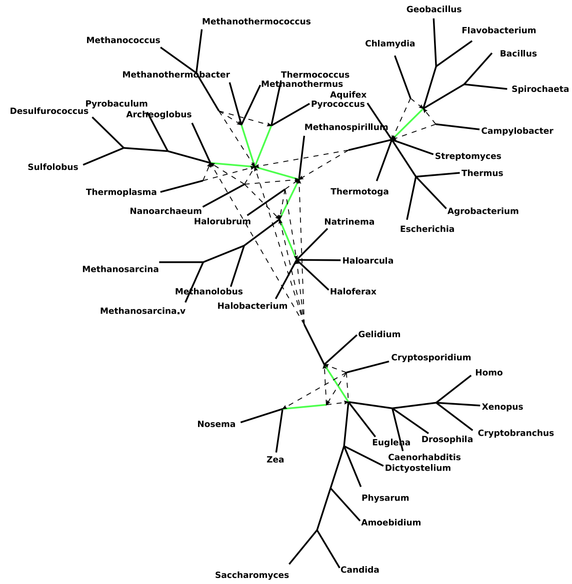
We now consider a real data example. The data consists of 48 5S rRNA sequence taken from across the Tree of Life. While the tree domains Eukarya, Archaea, and Eubacteria are well supported, the majority consensus tree finds little structure inside each domain at the 95% level.

9 Discussion

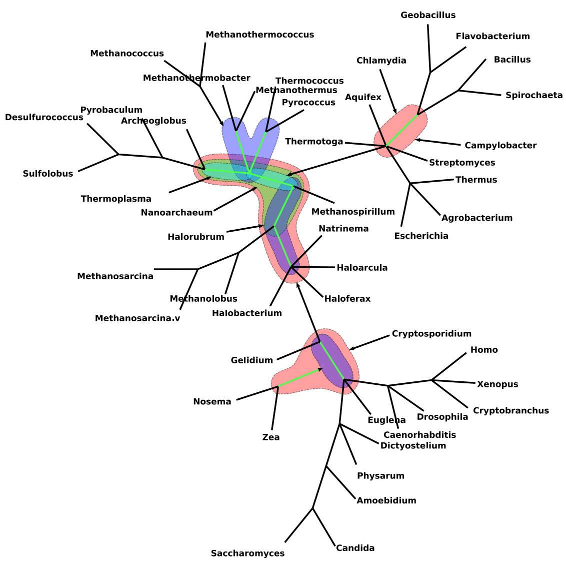
People usually display 50% consensus trees because there is in fact some supported structure that is invisible in 90% consensus trees. This structure is often visible in 90% extended consensus trees,



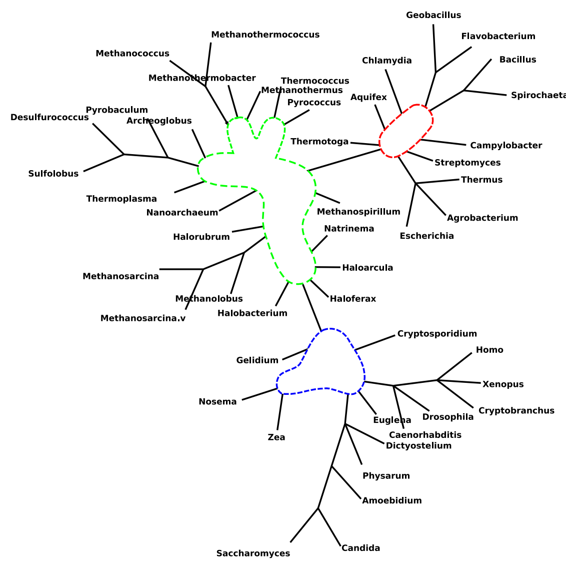
(a) 95% Majority Consensus



(b) Multiconnected-tree – Graph Representation



(c) Multiconnected-tree – "Cloud" Representation



(d) Polytomy Clouds

Figure 15: The extended majority consensus reveals structure inside polytomies. The phylogeny distribution is a posterior distribution for a phylogeny of 48 5S rRNA sequences from throughout the range of living organisms. (a) The majority consensus discovers only 15 full splits, out of a possible 45. (b) The extended majority consensus recovers an addition 9 partial splits, revealing hidden structure in the Archaea especially. (c) The cloud representation more clearly portrays attachment ranges and nested wandering relationships. (d) The boundaries of the clouds correspond to polytomies on the majority consensus tree.

though.

We suspect that researchers are inclined to use an M_l consensus tree with a small value of l in order to capture full splits with low support that imply partial splits with high support.

Other discussions of rogue taxa Thomson and Shaffer, Systematic Biology, early 2010 or late 2009.

http://treethinkers.blogspot.com/2010/02/going-rogue.html

Commonly suggested alternatives **[alternative #1]** It is commonly claimed that this could be simply handled by removing a few leaves, finding a highly supported tree on the remaining taxa, and then computing the probabilities for each attachment point. However, we take a different route, by focussing on supported phylogenetic hypothesis (partial splits) instead of on trees. Why?

[#alternative #2] By specifying nodes as an attachment point, we automatically allow attaching to every branch adjacent to the node. So, specifying that you could attach to branches would allow more flexibility... you could specify fewer trees.

[alternative #3] Instead of requiring that we attach to all nodes in between two possible attachment points, we could allow separate attachment points. This would allow more fine-tuning.

[reason #1] We are not trying to specify a set of trees, per se. We are trying to specify a set of (possibly partial) splits.

[reason #2] By focussing on the possible locations of some specific clade, we lose the ability to consider uncertainty for all remaining clades, since we must define locations on a *fixed* skeleton topology.

Cranston and Rannala's method might be further extended to allow partial splits that form a multiconnected tree. This would further decrease the necessity for pruning taxa.

Why do arrows connect to nodes rather than to branches? Because of the two-step resolution procedure, arrows may first be resolved to connect to a specific node; a new branch may then be inserted to reduce the resulting multifurcation, effectively allowing the arrow to attach to a branch adjacent to the node.

Partial Splits and the requirement to attach at every node between two attachment points As represented in figure 2, this makes the representation more lossy that it could, in theory, be. This is a direct result of requiring that information from the posterior be represented in terms of partial splits.

Interpreting Multiconnected Trees Biologists must correctly interpret consensus trees in order to them to be a helpful tool in summarizing posterior topology distributions. Therefore it is important that the figures are both easy to interpret, and not easily amenable to mistaken interpretations.

Majority consensus trees use multifurcating trees to represent a collection of full splits with *individually* high support. Therefore it is important to guard against misinterpreting majority consensus trees as containing splits that are *jointly* supported. Such an interpretation would indicate that the consensus tree is being used to represent a confidence set; this is explicitly not the approach that has been chosen. Likewise, extended majority consensus trees also represent a collection of full or partial splits that are individually supported.

Consensus networks can be difficult to interpret. They intentionally represent mutually incompatible splits, and some of these splits necessarily have low support (since, otherwise representing conflicting splits would not be necessary.) (Figure 5). However, we note that consensus networks can indeed represent splits that a multiconnected tree cannot.

Majority Consensus: Death by Short Branches The inability of majority consensus tree to represent uncertainty about attachment location is especially unfortunate for two reasons. First, any summary method that cannot handle short branches will eventually collapse as a sufficient number of taxa are added to the analysis. Secondly, breaking up long branches by the addition of new taxa is desirable to avoid systematic error due to model misspecification and the possible resulting long branch attraction. However, if the bias of long branch attraction is opposed by short branch confusion, then researchers may hesitate to add additional taxa, because the majority consensus would contain fewer branches even though the total amount of data has increased. Thus, the imperative to *improve* the estimate would be in conflict with the imperative to make a *visually representable* estimate.

"This implies that, as more and more taxa are added to a data set, the number of mutations supporting each full split on the new taxon set may decrease to the point where very few full splits are supported. In constrast, partial splits on the new taxon set can correspond to several subdivided branches by not specifying the location of some taxa."

Why not just make a probabilistic MAST? Q:You could pick leaves off the tree (or internal branches off the tree) until (a) the whole tree has probability greater than l or (b) all individual branches have probability greater than l.

A: Our goal is to avoid picking taxa off of the tree at all. In cases like Figure 6, this would remove half of the taxa from the tree. This problem would also occur when two clades reciprocally root each other, but with uncertainty in the root attachment points. In these case, and in many other cases, the issue isn't that particular branches are "wandering" or "rogue", but that we need to represent uncertainty. Furthermore, see argument in Section ?? that pruning taxa is undesirable.

Leaf Stability and other methods RadCon (Thorley and Page 2000) and Leaf stability indices (Thorley and Wilkinson 1999).

Talk with Stephen on this?

Criticism of consensus methods See response (to criticism of majority consensus methods in general) in Mark Holder's recent article in Sys Bio.

In support for consensus methods, mention the issue about how confidence cubes get smaller in each dimension as the number of dimensions grows. So, each part of the tree would lose resolution as more taxa are added in order to remain at some fixed bound (e.g. 95%) for the whole tree.

Median versus confidence interval.

Alternative: median and average distance? (e.g. Susan Holmes)

Future work (?): distances?

Parsimony: strict consensus implies that Σ *is* closed, and therefore... ?

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A Proofs

A.1 Definitions

- $\langle x \rangle$ refers to the set of all leaf-labelled binary trees compatible with x, whatever x is.
- If A is a ordered split, then A^t represents the same split but with the reverse orientation.
  - Therefore $\left(A^t\right)^t = A$.

- $\widehat{A}$ means the unordered version of the ordered split A.
- For some function f on ordered splits, $\widehat{f}$ means the projected version of that function for unordered splits.
  - This only makes sense if $f(C^t) = f(C)^t$
- $\check{\Sigma}$ means the ordered version of a set Σ of unordered splits[12]:
  - For each unordered split α in Σ, $\check{\Sigma}$ contains two splits A and A^t where $\widehat{A} = \widehat{A}^t = \alpha$.
- For a set $\mathcal{T}$ of trees, $q(\mathcal{T})$ refers to the set of quartets common to all the trees.
  - For a split A, $q(A)$ means all of the quartets implied by A. $q(A) = q(\langle A \rangle)$.
  - For a split set Σ, $q(\Sigma)$ means $\bigcup_{A \in \Sigma} q(A)$.

A.2 Basic facts about multiconnected trees

This section contains a number of lemmas that are used in later parts of the proof. Many of them concern how two splits may be related, given that they are both related to a third split in a certain way[13].

Lemma 74. *Given that Σ is a multiconnected tree and $A, B, C \in \check{\Sigma}$ and $A \downarrow B$ and $C < B$ and it is not the case that $A \downarrow C$, then $C < A^t$.*

Proof. We consider the 9 possible relationships between C and B that are given in section 4.4.

We first consider that A and C may be related through "$<$". We note that since $C < B$, C_1 must be a strict subset of B_1 by the definition of $<$; since $A \downarrow B$, B_1 is a strict subset of A_2. Therefore, C_1 must be a strict subset of A_2, and cannot intersect A_1. This rules out the possibilities $C < A$ and $A < C$ which require an overlap between C_1 and A_1. In addition, $A^t < C$ is ruled out since A_2 cannot be a strict subset of C_1.

We then consider that A and C may be related through "$\downarrow$". The premises explicitly rule out the possibility that $A \downarrow C$. We may also rule out the possibilities $A^t \downarrow C$ and $C \downarrow A$ because these possibility contradict the requirement that C_1 and A_2 must intersect; we rule out $C^t \downarrow A$ because C_2 contains B_2, which is in A_2.

Finally, A and C cannot be non-overlapping since C_1 is a subset of A_2.

Therefore, the only possible relationship between A and C is $C < A^t$. □

Lemma 75. *Given that Σ is a multiconnected tree and $A, B, C \in \check{\Sigma}$ and $A \Downarrow B$ and $C < B$ and it is not the case that $A \Downarrow C$, then $C < A^t$.*

Proof. By Lemma 74, either $A \downarrow C$ or $C < A^t$. We seek to show that if $A \downarrow C$ then $A \Downarrow C$. Therefore suppose that $A \downarrow C$, and let us additionally suppose that $A \not\Downarrow C$. Then there exists some $D \in \check{\Sigma}$ such that $A < D \downarrow C$. By Lemma 74, then either $D \downarrow B^t$ or $B^t < D^t$. In the first case, we have $A < D \downarrow B$, which contradicts $A \Downarrow B$. In the second case, we have $A < D < B$ which contradicts $A \downarrow B$. Since both of these cases are impossible, then we must have $A \downarrow C \implies A \Downarrow C$. □

Lemma 76. *If A, B, and D are distinct members of the multiconnected split set $\check{\Sigma}$, and $A < B$ and $A \downarrow D$ and then either*

[12] J:?

[13] Whole section: possibly replace "$A, B, C \in \check{\Sigma}$" with "$A$, B, and C, are distinct splits in $\check{\Sigma}$.

Case 1. $D < B$*, or*

Case 2. $D^t < B$*, or*

Case 3. $B \downarrow D$*, or*

Case 4. $B^t \downarrow D$*, or*

Case 5. $B \perp D$*.*

In case $B \perp D$*, there must be some* E *such* $E^t \downarrow B$ *and* $E \downarrow D$*. For any such* E*, we must have* $A < E$*.*

Proof. The following cases lead to contradictions, and therefore cannot occur:

- $B < D$: contradiction. $A < B < D \implies A < D$. This contradicts $A \downarrow D$.
- $B < D^t$: contradiction. $A < B < D^t \implies A < D^t$. This contradicts $A \downarrow D$.
- $D < B$: OK. (This is like in Lemma 74, but with the $A \downarrow C$ case used instead of disallowed.)
- $D^t < B$: OK. (Same as above w/ direction of D reversed, which doesn't matter.)
- $B \downarrow D$: OK. ($A < B \downarrow D$)
- $B^t \downarrow D$: OK. (Both A and B^t point down to D).
- $D \downarrow B$: contradiction. $A \downarrow D \downarrow B \implies A \downarrow B$. This contradicts $A < B$.
- $D^t \downarrow B$: contradiction. $A \downarrow D^t \downarrow B \implies A \downarrow B$. This contradicts $A < B$.
- $B \perp D$: OK.

  In this case, we must have some E such that $E \downarrow D$ and $E^t \downarrow B$.

  Now, $A_1 \subset B_1$ from $A < B$ and $B_1 \subset E_1$ from $E^t \downarrow B$. So, $A_1 \subset E_1$.

  Also $A \downarrow D$ and $E \downarrow D$, so $D_1 \cup D_2 \subseteq A_2$ and $D_1 \cup D_2 \subseteq E_2$, so $D_1 \cup D_2 \subseteq A_2 \cap E_2$.

  The fact that $\#(A, E) \geq 2$ and $A_1 \subseteq E_1$ an $A_2 \cap E_2 \neq \emptyset$ allows only the relationship $A < E$.

□

Lemma 77. *Given that* Σ *is a multiconnected tree and* $A, B, C \in \check{\Sigma}$ *and* $A \downarrow B$ *and* $C \downarrow B$ *then either*

Case 1: $A \downarrow C$*, or*
Case 2: $C \downarrow A$*, or*
Case 3: $A < C$*, or*
Case 4: $C < A$*, or*
Case 5: $C < A^t$*.*

We begin by noting that A_2 and C_2 must intersect, since both must contain $B_1 \cup B_2$. This rules out $A \perp C$, $A^t \downarrow C$, $C^t \downarrow A$, and $A^t < C$ from the nine possible relationships listed in section 4.4, leaving the five cases mentioned above.

Lemma 78. *Given that* Σ *is a multiconnected tree and* $A, B, C, D \in \check{\Sigma}$ *and* $A^t \downarrow C$ *and* $B^t \downarrow C$ *and* $A \downarrow D$ *and* $B \downarrow D$*, then either* $A < B$ *or* $B < A$*.*

Proof. By Lemma 77, $A^t \downarrow C$ and $B^t \downarrow C$ together imply that one of the following relationships between A and B must obtain.

- $A^t \downarrow B$
- $B^t \downarrow A$
- $A^t < B^t$ (which is the same as $B < A$)
- $B^t < A^t$ (which is the same as $A < B$)
- $B^t < A$

By the same Lemma, $A \downarrow D$ and $B \downarrow D$ together imply that A and B must simultaneously have one of these relationships: □

- $A \downarrow B$
- $B \downarrow A$
- $A < B$
- $B < A$
- $B < A^t$

The only relationships in the second list that are compatible with relationships in the first list are in fact $A < B$ and $B < A$. Therefore, one of these most hold.

Lemma 79. *Given that Σ is a multiconnected tree and $A, B, C \in \check{\Sigma}$ and $A \Downarrow B$ and $C \Downarrow B$, then $A_1 \cap C_1 = \emptyset$ and $A_1|C_1$ is implied by either A or C, or both.*

Proof. If $A_1|C_1$ is implied by any split, then $A_1 \cap C_1 = \emptyset$. Therefore we need only demonstrate the second claim of the lemma. So, considering the cases from Lemma 77:

Case 1: If $A \downarrow C$ then $A_1|C_1$ is implied by A since $C_1 \subset A_2$.

Case 2: If $C \downarrow A$ then $C_1|A_1$ is implied by C since $A_1 \subset C_2$.

Case 3: Not possible. If $A < C$ then ($A < C$ and $C \downarrow B$). But this contradicts the premise that A wanders *directly* over B.

Case 4: Not possible. If $C < A$ then ($C < A$ and $A \downarrow B$). But this contradicts the premise that C wanders *directly* over B.

Case 5: If $C < A^t$ then C implies $C_1|A_1$ since $A_1 \subset C_2$. Note that by symmetry, A also implies $A_1|C_1$, since $C < A^t$ can be rewritten as $A < C^t$. □

Lemma 80. *Given that Σ is a multiconnected tree and $A, B, C \in \check{\Sigma}$ and $A < B$ and $A < C$ then either*

Case i:$B < C$

Case ii: $C < B$

Case iii: $B^t < C$

Case iv: $B^t \downarrow C$

Case v: $C^t \downarrow B$.

Proof. Because $A_1 \subset B_1$ and $A_1 \subset C_1$, the intersection of B_1 and C_1 cannot be empty. This rules out $C < B^t$, $B^t \downarrow C$, $C^t \downarrow B$, and $B \perp C$ from the nine possible relationships listed in section 4.4, leaving the five cases mentioned above. □

Lemma 81. *Given that Σ is a multiconnected tree and $A, B, C, D \in \check{\Sigma}$ and $A < B < D$ and $A < C < D$ then either $B < C$ or $C < B$.*

Proof. Since $A < B$ and $A < C$, then by Lemma 80, we must have either

- $B < C$
- $C < B$
- $B^t < C$
- $B^t \downarrow C$
- $C^t \downarrow B$

Furthermore, we have $D^t < B^t$ and $D^t < C^t$, and so we must also have one of

- $B^t < C^t$ (which is the same as $C < B$)
- $C^t < B^t$ (which is the same as $B < C$)
- $B < C^t$
- $B \downarrow C$
- $C \downarrow B$

The only cases that occur in both lists are $B < C$ and $C < B$. □

Lemma 82. *The relation $<_o \equiv \{(A, B)\colon A \downarrow B \text{ or } A < B\}$ is a strict partial order on partial splits.*

Proof. Recall that $\downarrow$ and $<$ are themselves strict partial orders on splits (Sections 4.2.4 and 4.2.5). Since we cannot have either $A \downarrow A$ or $A < A$, we cannot have $A <_o A$ and so $<_o$ is irreflexive.

We note that $A < B$ rules out $B < A$ because $<$ is a partial order and rules out $B \downarrow A$ because $<$ and $\downarrow$ are mutually exclusive. Therefore, $A < B$ rules out $B <_o A$. Likewise, $A \downarrow B$ rules out $B \downarrow A$ because $\downarrow$ is a partial order and rules out $B < A$ because $<$ and $\downarrow$ are mutually exclusive. Therefore $A \downarrow B$ also rules out $B <_o A$. Merging these two results, we see that $A <_o B$ rules out $B <_o A$ and so $<_o$ is asymmetric.

In order to demonstrate that $<_o$ is transitive, we must prove four statements:

1. If $A \downarrow B$ and $B \downarrow C$ then $A <_o C$.
2. If $A < B$ and $B < C$ then $A <_o C$.
3. If $A < B$ and $B \downarrow C$ then $A <_o C$.
4. If $A \downarrow B$ and $B < C$ then $A <_o C$.

The first two statements follow from the transitivity of $\downarrow$ and $<$. To demonstrate the third statement we note that $C_1 \cup C_2 \subseteq B_2$ and $B_2 \subset A_2$. Therefore $C_1 \cup C_2 \subset A_2$ and so $A \downarrow C$. To prove the fourth statement, we consider two cases: $A \downarrow C$ and $A \not\downarrow C$. If $A \downarrow C$ then clearly $A <_o C$. If $A \not\downarrow C$ then we note that $A \downarrow B^t$ and $C^t < B^t$ and $A \not\downarrow C^t$; we may then apply Lemma 74 to conclude that $C^t < A^t$. This implies that $A < C$ and so $A <_o C$.

We now note that the statement "$A <_o B$ and $B <_o C$" is equivalent to ($A < B$ or $A \downarrow B$) and ($B < C$ or $B \downarrow C$). This in turn is equivalent to ($A \downarrow B$ and $B \downarrow C$) or ($A \downarrow B$ and $B <$

C) or ($A < B$ and $B \downarrow C$) or ($A < B$ and $B < C$). Because each of the four terms in this disjunction implies $A <_o C$, we may conclude that $A <_o B$ and $B <_o C$ implies $A <_o C$ and so $<_o$ is transitive.

Because $<_o$ is irreflexive, asymmetric, and transitive, it is a strict partial order. $\square$

The following lemma shows that all the branches that are adjacent to the right end of A are also adjacent to each other.

Lemma 83. *Given that Σ is a multiconnected tree and $A, B, C \in \check{\Sigma}$ and $A \lll B$ and $A \lll C$ then $B^t \lll C$.*

Proof. Because $A < B$ and $A < C$, we may reduce the possible relationship of B and C to the cases listed in Lemma 80. However, since $A \lll B$ and $A \lll C$, we may rule out $B < C$ and $C < B$. This leaves only $B^t < C$, $B^t \downarrow C$, and $C^t \downarrow B$. Since the last two cases are symmetrical, we assume that $B^t \downarrow C$ by way of contradiction. Since B^t does not directly wander, there must be some other split $D \in \check{\Sigma}$ such that $B^t < D$ and $D \downarrow C$. Now, we cannot have $D \downarrow A$, since this would imply $B^t \downarrow A$, which contradicts a premise. Therefore, $D < A^t$ by Lemma 74. However, this implies that $B^t < D < A^t$, which contradicts $A \lll B$. Therefore, $B^t \downarrow C$ leads to a contradiction and cannot occur; $C^t \downarrow B$ is ruled out by symmetry. This leaves only $B^t < C$.

Since $A \lll B$ and $A \lll C$, this demonstrates that B^t and C^t do not directly wander in Σ.

Now suppose that there is some $D \in \Sigma$ such that $B^t < D < C$. Since $D < C$ and $A < C$, Lemma 80 indicates that A and D must have one of the relationships $A < D$, $D < A$, $D < A^t$, $D \downarrow A$, or $A \downarrow D$. Since $D^t < B$ and $A < B$, we must also have one of $A < D^t$, $D < A^t$, $D^t < A^t$, $D^t \downarrow A$, or $A \downarrow D^t$. The cases that overlap are $A < D$, $D < A^t$, and $A \downarrow D$. We rule out $A < D$ because $D < C$, and $A < D < C$ contradicts $A \lll C$. We rule out $D < A^t$ because $D^t < B$ and $A < D^t < B$ contradicts $A \lll B$. Finally, we consider the possibility $A \downarrow D$. $A \downarrow D$ would require some $E \in \check{\Sigma}$ such that $A < E \downarrow D$. Since $E \not\downarrow B$ and $B^t < C$, Lemma 74 implies that $B^t < E^t$. But this would imply that $A < E < B$, which violates the premises. Therefore, there can be no D such that $B^t < D < C$, and so we may conclude that $B^t \lll C$. $\square$

The following lemma demonstrates that collections of adjacent branches with the $<_2$ relationship must form a linear structure that cannot branch.

Lemma 84. *Given that Σ is a multiconnected tree and $A, B \in \check{\Sigma}$ and $A <_2 B$ then there cannot be an ordered split C such that $A < C$ and $C^t < B$.*

Proof. If $A < C$ then $C_2 \subset A_2$. If $C^t < B$ then $C_2 \subset B_1$. Therefore if $A < C$ and $C^t < B$ then $A_2 \cap B_1$ is not empty. But this contradicts the premise $A <_2 B$, which requires that $A_2 \cap B_1$ is empty. $\square$

Lemma 85. *Given that Σ is a multiconnected tree and $A, B, C \in \check{\Sigma}$ and $A <_2 B$ and $A \downarrow C$ then either $B \downarrow C$ or $B \perp C$. If $B \perp C$ then there exists $D \in \Sigma$ such that $A < D^t$, $D \downarrow B$, and $D^t \downarrow C$.*

Proof. If $A \downarrow C$ then $C_1 \cup C_2 \subset A_2$. Since $A <_2 B$, we also have that $A_2 \cap B_1$ is empty. Since C_1 and C_2 are contained entirely within A_2, this means that C_1 and C_2 do not intersect B_1. Now since B and C are both part of a multiconnected tree, they must have a valid relationship from section 4.4. The only relationships that satisfy this condition are $B \downarrow C$ and $B \perp C$. Now, if $B \perp C$ in Σ, then there must exist some $D \in \check{\Sigma}$ such that $D \downarrow B$ and $D^t \downarrow C$. Since $B_1 \cup B_2 \subseteq D_2$ and B_1 intersects A_1 and B_2 intersects A_2, then D_2 intersects A_1 and A_2. Since $C_1 \cup C_2 \subseteq D_1$ and $C_1 \cup C_2 \subseteq A_2$, D_1 intersects A_2. Therefore $\#(A, D)$ is at least 3, and the only valid relationship with this form is $A < D^t$. $\square$

Lemma 86. *Given that Σ is a multiconnected tree and $A, B \in \check{\Sigma}$ and $A <_2 B$ then there is no split $C \in \Sigma$ such that $A \Downarrow C$.*

Proof. Suppose there was a split $C \in \Sigma$ such that $A \Downarrow C$. Then by Lemma 85, either $B \downarrow C$ or $B \perp C$. In the first case, $A < B \downarrow C$, and so it is not the case that $A \Downarrow C$ in Σ. In the second case, Lemma 85 indicates that there must exist a $D \in \Sigma$ such that $A < D^t \downarrow C$, and so it is not the case that $A \Downarrow C$ in Σ. Since both possibility lead to a contradiction, the lemma is proved. □

Lemma 87. *Given that Σ is a multiconnected tree and $A, B, C \in \check{\Sigma}$ and $B <_2 A$ and $C <_2 A$ then either $B <_2 C$ or $C <_2 B$.*

Proof. We first derive three points about B and C. The premises imply that $A_2 \subset B_2$ and $A_2 \subset C_2$ because $B < A$ and $C < A$. Therefore $B_2 \cap C_2 \neq \emptyset$; this is the first point.

Now, $B_2 \cap A_1 = \emptyset$ because $B <_2 A$ and $C_1 \subset A_1$ because $C < A$. Therefore $B_2 \cap C_1 = \emptyset$; this is the second point.

Now, by symmetry between B and C, we may conclude that $C_2 \cap B_1 = \emptyset$; this is the third point.

We consider the 9 possible relationships between B and C that are given in section 4.4.

Case 1. $B < C$ - We do not note any contraditions in this case.

Case 2. $C < B$ - We do not note any contraditions in this case.

Case 3. $B^t < C$ - This case contradicts points 1, 2 and 3.

Case 4. $C < B^t$ - This case contradicts points 2, and 3.

Case 5. $B \downarrow C$ - This case contradicts point 2.

Case 6. $B^t \downarrow C$ - This case contradicts points 1 and 3.

Case 7. $C \downarrow B$ - This case contradicts point 3.

Case 8. $C^t \downarrow B$ - This case contradicts points 1 and 2.

Case 9. $C \perp B$ - This case contradicts point 1.

Since B and C have a legal relationship in Σ, then this relationship must be either $B < C$ or $C < B$. Furthermore, since $B_2 \cap C_1 = \emptyset$ (point 2) and $C_2 \cap B_1 = \emptyset$ (point 3) we the relationship must be either $B <_2 C$ or $C <_2 B$. □

Lemma 88. *If Σ is a multiconnected tree containing A, B and C, and $A <_2 B$ and $A < C < B$, then $A <_2 C <_2 B$.*

Proof. This can be proved by demonstrating that $A_2 \cap C_1 = \emptyset$ and $C_2 \cap B_1 = \emptyset$. Now, $A <_2 B$ implies that $A_2 \cap B_1 = \emptyset$. Since $C_1 \subset B_1$, we have $A_2 \cap C_1 = \emptyset$. Because $C_2 \subset A_1$ we also have that $C_2 \cap B_1 = \emptyset$ and we are done. □

Lemma 89. *Given that A, B, C, and D are all distinct members of a multiconnected split set Σ, then if $A < B$ and $A < C$ and $B < D$ and $C < D$ then either $B < C$ or $C < B$.*

Proof. By Lemma 80, $A < B$ and $A < C$ implies that exactly one of the following must be true:

1. $B < C$

2. $C < B$

3. $B^t < C$

4. $B^t \downarrow C$

5. $C^t \downarrow B$

Because $D^t < B^t$ and $D^t < C^t$, the same lemma also implies that exactly one of the following must be true:

1. $B^t < C^t$, which is the same as $C < B$

2. $C^t < B^t$, which is the same as $B < C$

3. $B < C^t$

4. $B \downarrow C^t$

5. $C \downarrow B^t$

Now, the cases $B < C$ and $C < B$ occur in both groups; no other cases in the first group are compatible with any cases in the second group. Therefore, either $B < C$ or $C < B$ must be true. □

Lemma 90. *Given that A, B, C, and D are all distinct members of a multiconnected split set Σ, then if $B < A$ and $C \downarrow A$ and $B < D < C^t$ then either $B < D < A$ or $C < D^t \downarrow A$.*

Proof. By Lemma 76, $C < D^t$ and $C \downarrow A$ implies that exactly one of the following must be true:

1. $A < D^t$

2. $A^t < D^t$

3. $D^t \downarrow A$

4. $D \downarrow A$

5. $A \perp D$

Furthermore, by Lemma 80, $B < A$ and $B < C$ imply that exactly one of the following must be true:

1. $A < D$

2. $D < A$

3. $A^t < D$

4. $A^t \downarrow D$

5. $D^t \downarrow A$

The only cases that are occur in both groups are $D < A$ and $D^t \downarrow A$. □

Lemma 91. *Given that A, B, and C are all distinct members of a multiconnected split set Σ, then if $B \Downarrow A$ and $B \downarrow C$ and $C \downarrow A$ then $B \Downarrow C$.*

Proof. Suppose there were some $D \in \Sigma$ such that $B < D \downarrow C$. Then $B < D \downarrow C \downarrow A$, which contradicts $B \Downarrow A$. Thus, no such D exists, and $B \Downarrow C$. ☐

Lemma 92. *Given that A, B, C, and D are all distinct members of a multiconnected split set Σ, then if $B \Downarrow A$ and $B \Downarrow C$ and $C < D \downarrow A$ then $B \Downarrow D$.*

Proof. Now, $B \downarrow C$ and $C < D$ implies either $B \downarrow D$ or $B < D$ by 74. If $B < D$ then $B < D \downarrow A$, which contradicts $B \Downarrow A$. Therefore, it must be $B \downarrow D$. By Lemma91, $B \Downarrow D$. ☐

Lemma 93. *Given that A, B, C, and D are all distinct members of a multiconnected split set Σ, then if $A \downarrow B \downarrow C$ and $A < D \downarrow C$ then either $A < D \downarrow B \downarrow C$ or $A \downarrow B < D \downarrow C$.*

Proof. By Lemma 76 $A \downarrow B$ and $A < D$ together imply that $B < D$, $B^t < D$, $D \downarrow B$, $D^t \downarrow B$, or $B \perp D$. By Lemma 77, $B \downarrow C$ and $D \downarrow C$ together imply $B \downarrow D$, $D \downarrow B$, $B < D$, $D < B$, or $B < D^t$. The only possibilities on both lists are $B < D$ and $D \downarrow B$. ☐

Lemma 94. *Given that A, B, and C, all distinct members of a multiconnected split set Σ, then if $A \Downarrow B$ and $B \Downarrow C$ then $A \Downarrow C$.*

Proof. $A \downarrow C$ follows from the transitive property of $\downarrow$. By Lemma 93, if there is another split $D \in \Sigma$ such that $A < D \downarrow C$ then we must have either $A < D \downarrow B \downarrow C$ (which contradicts $A \Downarrow B$) or $A \downarrow B < D \downarrow C$ (which contradicts $B \Downarrow C$). Therefore there is no such split, and $A \Downarrow C$. ☐

A.3 Proof of compatibility

In this section, we show that a collection of partial splits that satisfies the criteria for being a multiconnected tree is also compatible.

Definition 95. Given that Σ is a multiconnected tree and A and B are ordered splits in $\check{\Sigma}$, then we define

$$\Lambda(A, B, \Sigma) \quad \equiv \quad \{C \in \check{\Sigma} : A \downarrow C \text{ and } C < B\}.$$

Definition 96. Given that Σ is a multiconnected tree and A and B are ordered splits in $\check{\Sigma}$, then we define

$$\Theta(A, B, \Sigma) \quad \equiv \quad \{C \in \check{\Sigma} : A \downarrow C \text{ and } C \downarrow B\}.$$

Definition 97. Given that Σ is a multiconnected tree and A and B are ordered splits in $\check{\Sigma}$, we define

$$\Gamma(A, B, \Sigma) \quad \equiv \quad \Lambda(A, B, \Sigma) \cup \Theta(A, B, \Sigma).$$

This set $\Gamma(A, B, \Sigma)$ can be thought of as the set of splits in Σ that need to be modified in order to attach A to the right side of B. We note that the splits are undered instead of unordered.

When a multiconnected tree $\check{\Sigma}$ has splits A and B such that $A \downarrow B$, then we seek to show that a new multiconnected tree can be found that results from resolving the split A to a specific side of B, so that it no longer wanders over B. We define a function $R_{A,B}(\cdot)$ to make a correspondence between splits in the original tree, and splits in the resolved tree.

Definition 98. For any ordered splits A and B we define a function $R_{A,B}(\cdot)$ on ordered splits such that for any ordered split C

$$R_{A,B}(C) \quad = \quad \begin{cases} B_1|A_1 \cup B_2 & \text{if } B_1|A_1 \cup B_2 \implies C \\ A_1 \cup B_2|B_1 & \text{if } A_1 \cup B_2|B_1 \implies C \\ C & \text{otherwise} \end{cases}$$

This function is intended to resolve the branch B by increasing its support $B_1 \cup B_2$ so that A is on the right side of $R_{A,B}(B)$. We note a few important facts about this function[14].

1. $R_{A,B}(B) = B_1|A_1 + B_2$ and $R_{A,B}(B^t) = A_1 + B_2|B_1 = R_{A,B}(B)^t$.

2. $R(R(C)) = R(C)$

3. $R(C) \implies C$ for all C.

   Since the only change that $R_{A,B}(C)$ can make is to add elements to C_1 or C_2, $R(C)$ must imply C.

4. $R(C) \neq C$ if and only if $R(C) = R(B)$ or $R(C) = R(B)^t$.
   Or equivalently, $R(C) = C$ if and only if $R(B)$ does not imply C and $R(C)^t$ does not imply C.

   The forward direction can be demonstrated by noting that $R(C) \neq C$ rules out the third case, so that one of $R(C) = R(B)$ or $R(C) = R(B)^t$ must be true. The reverse direction can be demonstrated by noting that $R(B) \implies B$ and so $\widehat{R(B)}$ cannot be in Σ and $R(C)$ cannot equal C. The equivalence holds because $R(C) = R(B)$ iff $R(B) \implies C$ and $R(C) = R(B)^t$ iff $R(B)^t \implies C$.
   Fixme: the reverse direction is not true in general... just on Σ if $A \downarrow B$ is in Σ.

5. $R^{-1}(X) = X$ unless $X = R(B)$ or $X = R(B)^t$.

   We seek to show that if the condition is false, then $R(X) = X$ and no other $Y \neq X$ has $R(Y) = X$. Assuming $X \neq R(B)$ and $X \neq R(B)^t$ then by point 4, $X = R(X)$. Also, for any Y, if $R(Y) = R(B)$ or $R(Y) = R(B)^t$ then $R(Y) \neq R(X)$ since $X \neq R(B)$ and $X \neq R(B)^t$. However, if $R(Y) \neq R(B)$ and $R(Y) \neq R(B)^t$, then $R(Y) = Y$, so if $Y \neq X$ then $R(Y) \neq R(X)$.

6. $R(C^t) = R(C)^t$.

   To see this, consider that $R(B^t) = R(B)^t$. For any split E that is not B, but is implied by $R_{A,B}(B)$ or $R_{A,B}(B)^t$, E^t is implied by $R_{A,B}(B)^t$, so that $R_{A,B}(E) = R_{A,B}(B)$ and $R_{A,B}(E^t) = R_{A,B}(B)^t$ and $R(E^t) = R(E)^t$. For any other split C, $R(C)^t = C^t = R(C)^t$.

Lemma 99. *Given that Σ is a multiconnected tree and $A, B, C \in \check{\Sigma}$ and $A \downarrow B$ and $R_{A,B}(C) = R_{A,B}(B)$ then either $B = C$ or $C <_2 B$. If $R_{A,B}(C) = R_{A,B}(B)^t$ then either $B = C^t$ or $C^t <_2 B$.*

Proof. Clearly, if $C = B$ then $R_{A,B}(C) = R_{A,B}(B)$. However, supposing that $C \neq B$, then $R_{A,B}(B) \implies C$ and $C_1 \subseteq B_1$ and $C_2 \subseteq B_2 + A_1$. The only legal relationships between B and C that satisfy $C_1 \subseteq B_1$ are $C < B$ and $B^t \downarrow C$. Now, if we assume that $B^t \downarrow C$ then $A \downarrow C$ because $A \downarrow B^t$. Therefore C_2 and A_1 do not intersect. However, if this is the case, then $C_2 \subseteq B_2 + A_1$ reduces to $C_2 \subseteq B_2$, and this contradicts our assumption. Therefore $C < B$. Furthermore, if

[14] Am I going to define "+" as being the union of disjoint sets, or am I just going to use $\cup$?

$R_{A,B}(B) \implies C$, then C_2 cannot intersect B_1 since it is a subset of $B_2 + A_1$, which does not interset B_1. Therefore $C_2 \cap B_1 = \emptyset$ and $C <_2 B$.

Finally, if $R_{A,B}(C) = R_{A,B}(B)^t$ then $R_{A,B}(C^t) = R_{A,B}(B)$ and we may apply the first part of the lemma to C^t instead of C. $\square$

Theorem 100. *Given that Σ is a multiconnected tree and $A \Downarrow B$ in Σ and $\Gamma(A,B,\Sigma)$ is empty then*

Claim 1: *The split set $\hat{R}_{A,B}(\Sigma)$ contains all splits in Σ except splits that are implied by $\hat{R}_{A,B}(B)$. In addition to these, it contains only the split $\hat{R}_{A,B}(B)$.*

Claim 2: *The split set $\hat{R}_{A,B}(\Sigma)$ is a multiconnected tree.*

Claim 3: *If $R^{-1}(R(B))$ contains more than one element then there is an ordered split $E \in \check{\Sigma}$ such that $R^{-1}(R(B)) = \{E, B\}$ and $R^{-1}(R(B)^t) = \{E^t, B^t\}$, where $E <_2 B$ and $E \ll B$ and $E_2 = B_2 + A_1$ so that $R(B) = B_1|E_2$.*

Claim 4: *For any ordered splits C and D in $\check{\Sigma}$.*

- *If $C < D$ then either $R(C) < R(D)$ or $R(C) = R(D)$.*
- *If $C \downarrow D$ then $R(C) \downarrow R(D)$ unless $R(D) \neq D$ and ($C = A$ or $C < A$ or $C < B$).*
  1. *If $D = B$ and ($C = A$ or $C < A$) then $R(C) < R(D)^t$.*
  2. *If $D = B^t$ and ($C = A$ or $C < A$) then $R(C) < R(D)$.*
  3. *If $D < B$ and $C < B$ then $R(C) < R(D)$*
  4. *If $D^t < B$ and $C < B$ then $R(C) < R(D)^t$*
- *If $C \perp D$ then $R(C) \perp R(D)$ unless ($R(D) \neq D$ and $A^t \downarrow C$) or ($R(C) \neq C$ and $A^t \downarrow D$)*
  1. *If $R(D) \neq D$ and $A^t \downarrow C$ then $R(D) \downarrow R(C)$ when $R(D) = R(B)$ and $R(D)^t \downarrow R(C)$ when $R(D) = R(B)^t$.*
  2. *If $R(C) \neq C$ and $A^t \downarrow D$ then $R(C) \downarrow R(D)$ when $R(C) = R(B)$ and $R(C)^t \downarrow R(D)$ when $R(C) = R(B)^t$.*

Proof. In this proof, for brevity, we will suppress the subscripts on the function $R_{A,B}$ and write R.

[claim 1] Clearly, every split in $\widehat{R}(\Sigma)$ is the image under $\widehat{R}$ of some split in Σ. For any split $C \in \Sigma$ that is not implied by $\widehat{R}(B)$, we have $\widehat{R}(C) = C$. Furthermore, for all other splits $C \in \Sigma$, we have $\widehat{R}(C) = \widehat{R}(B)$. This means that splits implied by $\widehat{R}(B)$ are not in $\widehat{R}(\Sigma)$, but are replaced with $\widehat{R}(B)$, which is present. Thus, claim 1 is demonstrated.

[claim 2] In order to show that $\hat{R}(\Sigma)$ is a multiconnected tree, we need to show that all pairs of splits in $\hat{R}(\Sigma)$ have a valid relationship. For any two splits C and D, $\hat{R}(C)$ and $\hat{R}(D)$ have a valid relationship if $\hat{R}$ leaves C and D unchanged. If both splits have been changed, then $\hat{R}(C) = \hat{R}(D)$ because both equal $\hat{R}(B)$, and so the splits are no longer distinct. This leaves case in which one of C or D is altered by $\hat{R}$, but the other one is not. Therefore, we consider ordered splits $C \in \check{\Sigma}$ such that $R(C) = C$ and seek to show that $R(C)$ and $R(B)$ maintain a valid relationship. Finally, we seek to show that for any two elements $C \perp D$ in $\check{\Sigma}$ where $R(C) \perp R(D)$, there remains an element X in $R(\check{\Sigma})$ such that $X \downarrow R(C)$ and $X \downarrow R(D)$.

First we note that $A < R(B^t)$ because $A_1 \subseteq A_1 + B_2$ and $B_1 \subset A_2$ and so A and $R(B)$ have a legal relationship in Σ'.

We now show that, for any ordered split $C \in \Sigma$ that is not A or A^t or B or B^t, either $\widehat{R(B)} \implies \widehat{C}$, or $R(B)$ and C have a legal relationship. We do this by considering the cases for how $R(B)$ may relate to C.

1. $C < B$ - We can conclude that $A \not\downarrow C$ from the fact that $\Gamma(A, B, \Sigma)$ is empty. Therefore, we can conclude $C < A^t$ by Lemma 74. Therefore $A_1 \subset C_2$ and so $B_2 + A_1 \subseteq C_2$. However, there are two cases:

   Case i: $B_2 + A_1 \subset C_2$ - If the subset is a proper subset then $C < R(B)$.

   Case ii: $B_2 + A_1 = C_2$ - If the subset is not a proper subset then $R(B) \implies C$ since $C_1 \subset B_1$; therefore $R(C) = R(B)$. We note that $C_1 \subset B_1$ and $C_2 = B_2 + A_1$ and since $R(B)$ is a split. This implies that $C_2 \cap B_1$ must be empty. Since $C_1 \subset B_1$ and $B_2 \subset C_2$, and $C_2 \cap B_1$ is empty, we must have $C <_2 B$ in this case.

2. $B < C$ - If $B_1|B_2 < C_1|C_2$ then $B_1|B_2 + A_1 < C_1|C_2$ by the definition of "$<$". To determine the relationship of C to A, we can conclude that either $A \downarrow C$ or $A < C$ by Lemma 74.

3. $C^t < B$ - Case 1 applies here by reversing C.

4. $B < C^t$ - Case 2 applies here by reversing C.

5. $C \downarrow B$ - Lemma 77 applies. Therefore we consider the 5 cases for how C may relate to A:

   Case i: $A \downarrow C$ - Cannot occur because $\Gamma(A, B, \Sigma)$ is empty.

   Case ii: $C \downarrow A$ - Since $C \downarrow A$, then C_2 contains A_1. Also $C \downarrow A \downarrow B$ and so $C \downarrow B$. Since $C \downarrow B_1|B_2$ and $A_1 \subset C_2$, then $C \downarrow B_1|A_1 + B_2$.

   Case iii: $A < C$ - Cannot occur because $\Gamma(A, B, \Sigma)$ is empty.

   Case iv: $C < A$ - We have $C < A \downarrow B$ and so $C \downarrow B$. However, after refining B to $B_1|B_2 + A_1$, we still have $A_1|A_2 < A_1 + B_2|B_1$. Therefore, $C < A < R(B)^t$ and $C < R(B)^t$.

   Case v: $A < C^t$ - Because $A_1 \subset C_2$, then $C \downarrow B_1|B_2 + A_1$ if $C \downarrow B$.

6. $C^t \downarrow B$ - Case 5 applies here by reversing C.

7. $B \downarrow C$ - If $B_1|B_2 \downarrow C$, then $B_1|A_1 + B_2 \downarrow C$ by the definition of "$\downarrow$". We note that $A \downarrow C$.

8. $B^t \downarrow C$ - If $B_2|B_1 \downarrow C$, then $B_2 + A_1|B_1 \downarrow C$ by the definition of "$\downarrow$". We note that $A \downarrow C$.

9. $B \perp C$ - There must be some $D \in \check{\Sigma}$ such that $D \downarrow B$ and $D^t \downarrow C$. We may therefore apply case 5 to $D \downarrow B$, considering the three valid subcases:

   Case i: If $D \downarrow A$ (case 5.ii) then $D \downarrow R(B)$ and therefore (i) $C \perp R(B)$ and (ii) $D \downarrow R(B)$ while $D^t \downarrow C$. Also, $A \perp C$.

   Case ii: If $D < A$ (case 5.iv) then $D < R(B)^t$. Since $R(B) < D^t \downarrow C$ we have $R(B) \downarrow C$. Also, $A^t < D^t \downarrow C$ and so $A^t \downarrow C$.

   Case iii: If $A < D^t$ (case 5.v) then $D \downarrow R(B)$ and therefore (i) $C \perp R(B)$ and (ii) $D \downarrow R(B)$ while $D^t \downarrow C$. Also $A < D^t \downarrow C$ and so $A \downarrow C$.

Finally, we seek to show that for any splits C, D, and E in Σ , if $E \downarrow C$ and $E^t \downarrow D$ and $R(C)$ and $R(D)$ are non-overlapping then $R(E) \downarrow R(C)$ and $R(E) \downarrow R(D)$. Now we must have either $R(E) \downarrow R(C)$ or $R(E) < R(C)$ or $R(E) < R(C)^t$, and we must have either $R(E)^t \downarrow R(D)$ or $R(E)^t < R(D)$ or $R(E)^t < R(D)^t$. Now if $R(E) < R(C)$ then $R(C)^t < R(E)^t$. Considering each of the cases for how E and D relate leads to $R(C)^t < R(E)^t \downarrow R(D)$ or $R(C)^t < R(E)^t < R(D)$ or $R(C)^t < R(E) < R(D)^t$. Each of these cases contradicts the premise that $R(C)$ and $R(D)$ are non-overlapping. Likewise considering $R(E) < R(C)^t$ leads to the same conclusion. Therefore, the only possibility remaining is $R(E) \downarrow R(C)$. But, but symmetry, we must have $R(E)^t \downarrow R(D)$. Therefore claim 2 is demonstrated.

[claim 3] We also note that the only case in which a split $C \neq B$ has $R(C) = R(B)$ is in case 1.ii, and in this case $C_2 = B_2 + A_1$ and $C <_2 B$. Therefore for any $C \in R^{-1}(R(B))$ where $C \neq B$, $C_2 = B_2 + A_1$ and $C <_2 B$. The only other case in which a split C has $\widehat{R(C)} = \widehat{R(B)}$ is case 3, in which the results of case 1.ii apply to C^t. Therefore, without loss of generality, when considering some split C such that $\widehat{C} \neq \widehat{B}$ but $\widehat{R(C)} = \widehat{R(B)}$, we can restrict ourselves to the case where $C < B$ and $R(C) = R(B)$.

Now imagine that there are two distinct splits C and D besides B such that $C < B$ and $D < B$ and $R(C) = R(D) = R(B)$. In that case, both $C_2 = B_2 + A_1$ and $D_2 = B_2 + A_1$ so that $C_2 = D_2$. However, none of the possible relationships between C and D in section 4.4 would allow $C_2 = D_2$. Therefore, at most one split C besides B has $R(C) = R(B)$.

Now, imagine that $R^{-1}(R(B)) = \{C, B\}$ but there is an ordered split $D \in \check{\Sigma}$ such that $C < D < B$. Then $D_1 \subset B_1$ because $D < B$ and $D_2 \subset C_2$ because $C < D$. But, then $D_1 \subset R(B)_1$ and $D_2 \subset R(B)_2$ because $C_2 = R(B)_2$, and so $R(B) \implies D$. However, this contradicts the fact that at most one split C in $\check{\Sigma}$ besides B can be implied by $R(B)$, and so we may conclude that such a split D does not exist in $\check{\Sigma}$. Furthermore, such a split C cannot directly wander over any split in $\check{\Sigma}$ by Lemma 86. Therefore, if there is a split C in $\check{\Sigma}$ such that $C \neq B$ and $R(B) \implies C$ then $C \ll B$, and we have demonstrated the second claim.

[claim 4i] Consider ordered splits $C, D \in \check{\Sigma}$. Now if $C < D$ then if neither C or D are changed by $R(\cdot)$ then $R(C) < R(D)$, and claim 4.i holds. Also, if C and D are both changed, then $R(C) = R(D) = R(B)$ or $R(C) = R(D) = R(B)^t$ by claim 3, and claim 4.i holds. Finally, suppose only one of C or D is changed. Let us suppose that it is C that is changed, but that $R(D) = D$. Now, if $C = B$ or $C = B^t$, then cases 1-4 of claim 1 show that $C < D$ implies that $R(C) < R(D)$, since $R(C) \neq R(D)$. However, if C does not equal B or B^t then by claim 3, there is exactly one other ordered split $E \in \check{\Sigma}$ such that $R(E) = R(B)$, and for this split $E <_2 B$; C may equal E or E^t. Now suppose $C = E$ so that $E < D$, and let us consider the relationship of D to B. By claim 3, we may not have $D < B$. By Lemma 84 we may not have $D^t < B$. Now if $B^t \downarrow D$, we must have either $E^t \downarrow D$ or $E \perp D$ by Lemma 85 but this contradicts $E < D$. By Lemma 80 this leaves only the possibilities $B < D$ and $D^t \downarrow B$. Now, we now that $D_2 \subset E_2$ because $E < D$. So if $B < D$ then $B_1 \subset D_1$ too, and so $R(E) = B_1|E_2 < D$. Alternatively, if $D^t \downarrow B$, then $B_1 \subset D_1$ and so again $R(E) = B_1|E_2 < D$. Therefore if $E < D$, then $R(E) < D$. Finally, let us consider whether $R(E)^t < D$ whenever $E^t < D$. We must show that that $E_2 \subset D_1$ and $D_2 \subset B_1$. Now $E^t < D$ directly implies that $E_2 \subset D_1$ and it implies that $D_2 \subset E_1$. Since $E < B$ we have $E_1 \subset B_1$, so that $D_2 \subset E_1 \subset B_1$. Therefore for any $C \in \check{\Sigma}$, if $C < D$, then $R(C) < R(D)$ when $R(C) = R(B)$ and $R(D) = D$. In order to consider the case when $C < D$ and $R(C) = C$ and $R(D) \neq D$, we let $C' = D^t$ and $D' = C^t$ and apply the previous case to claim that when $C' < D'$ and $R(C') \neq C'$ and $R(D') = D'$ then $R(C') < R(D')$. Then $D^t < C^t$ implies that $R(D)^t < R(C)^t$ and we are done.

[claim 4.ii] Consider splits $C, D \in \check{\Sigma}$ where $C \downarrow D$. Clearly, if $R(C) = C$ and $R(D) = D$ then $R(C) \downarrow R(D)$. Also, C and D cannot both be changed by $R(\cdot)$ according to claim 3. Therefore, we must consider the two cases $R(C) \neq C \wedge R(D) = D$ and $R(C) = C \wedge R(D) \neq D$. We note that if $C \downarrow D$ then $R(C) \downarrow D$, since $R(C) \implies C$. So, $C \downarrow D$ implies that $R(C) \downarrow R(D)$ in this case. We therefore consider the final case $R(C) = C \wedge R(D) \neq D$. Now, we must have either $D = B$ or $D = B^t$ or $D = E$ or $D = E^t$ for some split E such that $E <_2 B$. If $D = B$ then if $C = A$ then $C < R(D)^t$. Likewise if $C \neq A$ but $C < A$ then case 5.iv implies that $C < R(D)^t$. However, case 5 implies that if $C \neq A$ and $C \not< A$ then $C \downarrow R(D)$. By applying this result to $D^t = B$ we find that $C \downarrow R(D)$ unless $C = A$ or $C < A$, and in this case $R(C) < R(D)$.

Now if $D = E$, then either $C \downarrow B$ or $C \not\downarrow B$. If $C \not\downarrow B$ then $C < B$ by Lemma 74, and $C_1 \subset B_1$. Additionally, $E_2 \subset C_2$ because $C \downarrow E$ and so we have $C < B_1|E_2$. Since $R(C) = C$ and $R(D) = R(B) = B_1|E_2$, in this case we have $R(C) < R(D)$. However, if $C \downarrow B$ then we begin by noting that $E_1 \cup E_2 \subset C_2$. Since $E_2 = B_2 + A_1$, we have $B_2 \cup A_1 \subset C_2$. Since $C \downarrow B$, we also have $B_1 \cup B_2 \subset C_2$. Therefore $B_1 \cup B_2 \cup A_2 \subset C_2$ and $C \downarrow R(B)$. Therefore, $R(C) \downarrow R(D)$. By applying this result to $D^t = E$ we find that in this case $C \downarrow R(D)$ unless $C < B$, in which case $R(C) < R(D)^t$. To express this in the terms of claim 4, we note that $D = E$ is equivalent to $R(D) = R(B)$ and $D < B$; $D = E^t$ is equivalent to $R(D) = R(B)$ and $D^t < B$. Therefore claim 4.2 is demonstrated.

[claim 4.iii] We now consider splits C and D where $C \perp D$. Now, if $R(C) = C$ and $R(D) = D$ then $R(C) \perp R(D)$ in Σ by claim 2. The case $R(C) \neq C$ and $R(D) \neq D$ cannot occur, since $R(C)$ and $R(D)$ would have to be $R(B)$ or $R(B)^t$ and would therefore have overlapping taxon sets. Therefore, suppose that $R(C) = C$ and $R(D) \neq D$. Now by case 9 of claim 2, $R(C) \perp R(D)$ unless $A^t \downarrow C$, in which case $R(D) \downarrow C$ if $R(D) = R(B)$ and $R(D)^t \downarrow C$ if $R(D) = R(B)^t$. The second part of claim 4.iii follows by the symmetry of C and D. ☐

Theorem 101. *Given that $A \Downarrow B$ in Σ so that $\widehat{R}_{A,B}(\Sigma)$ is a multiconnected tree, then if $X \Downarrow Y$ in $\widehat{R}_{A,B}(\Sigma)$ then there exists $C, D \in \check{\Sigma}$ such that $R_{A,B}(C) = X$ and $R_{A,B}(D) = Y$ and $C \Downarrow D$ in Σ. This means that for any $C, D \in \Sigma$ where $C \Downarrow D$, $R_{A,B}(C) \Downarrow R_{A,B}(D)$ unless $R_{A,B}(C) \neq C$, and in this case, the other split E such that $R(E) = R(B)$ has $R(E) \Downarrow R(D)$.*

Hmm.. should I add that $C \Downarrow D$ implies $R(C) \Downarrow R(D)$ unless $R(C) = R(B)$?

Proof. In this proof, for brevity, we will suppress the subscripts on the function $R_{A,B}$ and write R.

We first consider any elements $C \in R^{-1}(X)$ and $D \in R^{-1}(Y)$ must relate to each other given that $X \Downarrow Y$. Since $X \downarrow Y$ then by claim 4 of theorem 100, we must have either $C \downarrow D$ or $C \perp D$ in Σ. If $C \perp D$ but $R(C) \downarrow R(D)$ then $A^t \downarrow D$ and $R(D) = D$ and $R(C) = R(B)$. Also $A < R(B)^t$ so $R(B) < A^t$. So, $R(B) < A^t \downarrow D$. Now $X = R(B)$ and $Y = D$ so we have $X < A^t \downarrow Y$; this contradicts the assumption that $X \Downarrow Y$, ruling out the possibility $C \perp D$. Thus if $X \Downarrow Y$ then $C \downarrow D$ for any $C \in R^{-1}(X)$ and $D \in R^{-1}(Y)$.

We now seek to find specific ordered splits $C \in R^{-1}(X)$ and $D \in R^{-1}(Y)$ such that $C \Downarrow D$. In order to do this, we first construct a specific split $C \in R^{-1}(X)$ such that no other split $E \in R^{-1}(X)$ has $C < E$. If $R^{-1}(X)$ contains only one split then we set C equal to this split, and the condition on C is trivially satisfied since there is no other split $E \in R^{-1}(X)$. By claim 3 of theorem 100, then if $R^{-1}(X)$ contains more than one split, then $R^{-1}(X)$ must equal $\{F, B\}$ or $\{F^t, B^t\}$ for some $F \in \check{\Sigma}$ such that $F \ll_2 B$. Thus, we may simply choose the right-most of the two splits in the set. In the first case, we select $C = B$; all other elements of $R^{-1}(X)$ are to the left of B since $F \ll_2 B$. In the second case, we select $C = F^t$; all other elements of $R^{-1}(X)$ are to the left of F^t since $B^t \ll_2 F^t$. Thus we have demonstrated by construction such a C exists.

Now, using this specific C, suppose that there exists a split $E \in \check{\Sigma}$ such that $C < E \downarrow D$ in Σ. Because of the way we have constructed C, we can rule out the possibility that $R(C) = R(E)$. Additionally, we cannot have $R(E) = R(D)$ because we this is not possible if $E \downarrow D$. Therefore, $R(C)$, $R(E)$, and $R(D)$ are all distinct. Now, if $C < E$ and $R(C) \neq R(E)$ then $R(C) < R(E)$ by claim 4 of theorem 100. Also, if $E \downarrow D$ then either $R(E) < R(D)$ or $R(E) < R(D)^t$ or $R(E) \downarrow R(D)$ by claim 4 theorem 100. However, given that $R(C) < R(E)$, the first two possibilities imply that $X < Y$ or $X < Y^t$ which contradict the premises. Therefore, if $C < E \downarrow D$, then $R(C) < R(E) \downarrow R(D)$. Therefore, there exists a split F in $R(\check{\Sigma})$ such that $X < F \downarrow Y$. Taking the contrapositive of this, we see that if there does not exist a split F in $R(\check{\Sigma})$ such that $X < F \downarrow Y$, then there does not exist a split E in $\check{\Sigma}$ such that $C < E \downarrow D$. Therefore, if $X \Downarrow Y$ in Σ', then

$C \Downarrow D$ in Σ. Therefore, if $X \Downarrow Y$ in $R(\check{\Sigma})$ there exists C and D in $\check{\Sigma}$ such that $R(C) = X$ and $R(D) = Y$ and $C \Downarrow D$ in Σ. $\square$

Hmm... if we simply seek a function that increases with $R(\cdot)$ then $|q(\cdot)|$ would be one. But (i) this is different than decreasing (ii) to a known bound that implies a multifurcating tree.

Definition 102. We define the set $W(\Sigma)$ on a multiconnected tree Σ as the set

$$W(\Sigma) \equiv \left\{(A, B) \in \check{\Sigma} \times \check{\Sigma} : A \Downarrow B \text{ in } \Sigma\right\}$$

We refer to this as the set of wandering pairs.

Lemma 103. *If $A \Downarrow B$ in Σ so that $\widehat{R}_{A,B}(\Sigma)$ is a multiconnected tree, then $|W(\Sigma)| > \left|W(\widehat{R}_{A,B}(\Sigma))\right|$.*

Proof. In this proof, we will suppress the subscripts on the function $R_{A,B}$ for brevity and just write R.

We define the function $R \times R : \check{\Sigma} \times \check{\Sigma} \to R(\check{\Sigma}) \times R(\check{\Sigma})$ as taking any pair of ordered splits (C, D) to its image $(R(C), R(D))$. We note that each point y in $R(\check{\Sigma}) \times R(\check{\Sigma})$ is associated with a set of points $[R \times R]^{-1}(y)$ in $\check{\Sigma} \times \check{\Sigma}$; furthermore, these subsets are non-overlapping and cover $\check{\Sigma} \times \check{\Sigma}$. Therefore we can decompose $W(\Sigma)$ into partitions of the form $[R \times R]^{-1}(y)$.

Now, the function $R \times R$ maps a number of partitions in $W(\Sigma)$ to $W(\widehat{R}(\Sigma))$, whereas some fall outside of $W(\widehat{R}(\Sigma))$. Now for each pair y in $W(\widehat{R}(\Sigma))$, there must be some point in $[R \times R]^{-1}(y)$ that is also in $W(\Sigma)$. Therefore, the number of partitions in $W(\Sigma)$ that map to $W(\widehat{R}(\Sigma))$ must be at least $\left|W(\widehat{R}(\Sigma))\right|$. Furthermore, the pair (A, B) is present in $W(\Sigma)$, but $[R \times R](A, B) = (A, R(B))$ and $(A, R(B))$ is not a member of $W(\widehat{R}(\Sigma))$ since $A < R(B)^t$. Therefore, in addition to the partitions that map to $\left|W(\widehat{R}(\Sigma))\right|$, at least one more is present in $W(\Sigma)$. We may therefore conclude that $|W(\Sigma)| \geq \left|W(\widehat{R}_{A,B}(\Sigma))\right| + 1$, which concludes the proof. $\square$

Lemma 104. *Given that Σ is a multiconnected tree containing the distinct ordered splits A, B, and C and that $A \downarrow B$ and $A \downarrow C$ then for any ordered split $D \in \check{\Sigma}$*

- *$R_{A,B}(B) \implies D$ and $R_{A,C}(C) \implies D$ cannot both be true.*
- *$R_{A,B}(B) \implies D$ and $R_{A,C}(C)^t \implies D$ cannot both be true.*

Proof. We first assume that $R_{A,B}(B) \implies D$ and $R_{A,C}(C) \implies D$ for some ordered split D by way of contradiction. We note that $D <_2 B$ and $D <_2 C$ by Lemma 99. Therefore, $B_2 \subset D_2$ and $C_2 \subset D_2$. Secondly, by the assumption we also have $D_2 \subseteq B_2 + A_1$ and $D_2 \subseteq C_2 + A_1$. Now, since $A \downarrow B$, we know that A_1 and B_2 are disjoint, as are A_1 and C_2. Therefore $B_2 = D_2 \cap A^C$. But also $C_2 = D_2 \cap A^C$, and so $B_2 = C_2$. This contradicts the premise that B and C have a legal relationship in Σ. Therefore, there is no split D such that $R_{A,B}(B) \implies D$ and $R_{AC}(C) \implies C$.

We now assume that $R_{A,B}(B) \implies D$ and $R_{A,C}(C)^t \implies D$ by way of contradiction. This implies that $D_1 \subseteq B_1$ and $D_1 \subseteq C_2 + A_1$ and so $D_1 \subseteq B_1 \cap (C_2 + A_1)$. It also implies that $D_2 \subseteq B_2 + A_1$ and $D_2 \subseteq C_1$ and so $D_2 \subseteq (B_2 + A_1) \cap C_1$. Since A_1 is disjoint from B and C, this implies that $D_1 \subseteq B_1 \cap C_2$ and $D_2 \subseteq B_2 \cap C_1$. But this contradicts the premise that D has a legal relationship with B. $\square$

Definition 105. Given that Σ is a multiconnected tree containing ordered splits A and B and $A \Downarrow B$ in Σ, and given that Φ is a subset of Σ such that A wanders over every split in Φ, then we define the function $U_{A,\Phi}$ with range $\check{\Sigma}$ such that for any ordered split $D \in \check{\Sigma}$:

Should we write $U_{A,\Phi,\Sigma}$? That is, should we include the Σ?

$$U_{A,\Phi}(D) \;\; \equiv \;\; \begin{cases} R_{A,C}(C) & \text{if } [\exists C \in \Phi] \; R_{A,C}(C) \implies D \\ R_{A,C}(C)^t & \text{if } [\exists C \in \Phi] \; R_{A,C}(C)^t \implies D^t \\ D & \text{otherwise} \end{cases}$$

By Lemma 104, the different cases are mutually exclusive.

Lemma 106. *Given that Σ is a multiconnected tree containing ordered splits A and B and $A \Downarrow B$ in Σ, and given that Φ is a subset of Σ such that A wanders over every split in Φ, then if $\{C_i\}_{i=1}^n$ is any ordering of splits in Φ then*

$$U_{A,\Phi} \;\; = \;\; R_{A,C_1} \circ R_{A,C_2} \circ \ldots \circ R_{A,C_n}$$

Proof. We examine each case in the definition of $U_{A,\Phi}$ to show that $R_{A,C_1} \circ R_{A,C_2} \circ \ldots \circ R_{A,C_n}$ yields the same result. First, for any ordered split D, if there is no split $C \in \Phi$ such that $R_{A,C}(C) \implies D$ or $R_{A,C}(C)^t \implies D$ then $R_{A,C_i}(D) = D$ for all i and so $U_{A,\Phi}(D) = R_{A,C_1} \circ R_{A,C_2} \circ \ldots \circ R_{A,C_n}(D) = D$.

Second, let us consider the case where there exists some $C_i \in \Phi$ such that $R_{A,C_i}(C_i) \implies D$ or $R_{A,C_i}(C)^t \implies D$. We may write:

$$R_{A,C_1} \circ R_{A,C_2} \circ \ldots \circ R_{A,C_n}(D) \;\; = \;\; R_{A,C_1} \circ R_{A,C_2} \circ R_{A,C_{i-1}}(R_{A,C_i}(R_{A,C_{i+1}} \ldots \circ R_{A,C_n}(D))).$$

Now, by Lemma 104, for any $C_j \in \Phi$ such that $i \neq j$, $R_{A,C_j}(D) = D$, and so

$$R_{A,C_1} \circ R_{A,C_2} \circ \ldots \circ R_{A,C_n}(D) \;\; = \;\; R_{A,C_1} \circ R_{A,C_2} \circ R_{A,C_{i-1}}(R_{A,C_i}(D)).$$

Furthermore, we cannot have $R_{A,C_j}(C_j) \implies R_{A,C_i}(C_i)$ because this would require that $R_{A,C_j}(C_j) \implies C_i$ and $R_{A,C_i}(C_i) \implies C_i$; likewise we cannot have $R_{A,C_j}(C_j) \implies R_{A,C_i}(C_i)^t$ because this would require that $R_{A,C_j}(C_j) \implies C_i^t$ and $R_{A,C_i}(C_i)^t \implies C_i^t$. Therefore $R_{A,C_j}(R_{A,C_i}(C_i)) = R_{A,C_i}(C_i)$ and $R_{A,C_j}(R_{A,C_i}(C_i)^t) = R_{A,C_i}(C_i)^t$. This leads to the conclusion that in if $R_{A,C_i}(C_i) \implies D$ or $R_{A,C_i}(C_i)^t \implies D$ then

$$R_{A,C_1} \circ R_{A,C_2} \circ \ldots \circ R_{A,C_n}(D) \;\; = \;\; R_{A,C_i}(D) = U_{A,\Phi}(D).$$

This covers all the cases in the definition of $U_{A,\Phi}$. Since $U_{A,\Phi}(D) = R_{A,C_1} \circ R_{A,C_2} \circ \ldots \circ R_{A,C_n}(D)$ for all D, the Lemma is proved. □

Definition 107. Given that Σ is a multiconnected tree containing ordered splits A and B and $A \Downarrow B$ in Σ, and given that Φ is a subset of Σ such that A wanders over every split in Φ, then we define the function $S_{A,B,\Sigma}$ with range $\check{\Sigma}$ such that for any ordered split $D \in \check{\Sigma}$:

$$S_{A,B,\Sigma}(D) \;\; \equiv \;\; U_{A,\{B\} \cup \Gamma(A,B,\Sigma)}$$

Definition 108. Given that Σ is a multiconnected tree and $A \Downarrow B$ in Σ, we define $M_{A,B,\Sigma}$ as

$$M_{A,B,\Sigma} \;\; = \;\; \left(\bigcup_{C \in B \cup \Gamma(A,B,\Sigma)} \widehat{R}_{A,C}(C) \right).$$

Lemma 109. *Given that Σ is a multiconnected tree and $A \Downarrow B$ in Σ, we may write $\widehat{S}_{A,B,\Sigma}(\Sigma)$ as*

$$\widehat{S}_{A,B,\Sigma}(\Sigma) \quad = \quad \{C \in \Sigma : \neg M_{A,B,\Sigma} \implies C\} \cup M_{A,B,\Sigma}.$$

Proof. proof...!!! □

Theorem 110. *Given that Σ is a multiconnected tree and $A, B \in \check{\Sigma}$ and $A \Downarrow B$ in Σ, then $\widehat{S}_{A,B,\Sigma}(\Sigma)$ is a multiconnected tree in which $S_{A,B,\Sigma}(A) < S_{A,B,\Sigma}(B)^t$. Also, $|W(\widehat{S}_{A,B,\Sigma}(\Sigma)| < |W(\Sigma)|$*

Proof. We first seek to show that A directly wanders over every ordered split in $\Gamma(A, B, \Sigma)$. Consider an ordered split $C \in \Theta(A, B, \Sigma)$. If we assume that A does not directly wander over C, then there is some $D \in \check{\Sigma}$ such that $A < D$ and $D \downarrow C$. But then $D \downarrow B$ because $C \downarrow B$ and since $A < D$ this contradicts $A \Downarrow B$. Therefore, $A \Downarrow C$ for $C \in \Theta(A, B, \Sigma)$. Next consider a split $C \in \Lambda(A, B, \Sigma)$. If A does not directly wander over C, then there is some $D \in \check{\Sigma}$ such that $A < D$ and $D \downarrow C$. Now, by Lemma 74, either $D \downarrow B$ or $D < B$. If $D \downarrow B$ then we would have $A < D \downarrow B$, which contradicts the premise $A \Downarrow B$. If $D < B$ then we would have $A < D < B$ which also contradicts the premise. Therefore, $A \Downarrow C$ for $C \in \Lambda(A, B, \Sigma)$. Since $\Gamma(A, B, \Sigma) = \Lambda(A, B, \Sigma) \cup \Theta(A, B, \Sigma)$, A must also directly wander over every element of $\Gamma(A, B, \Sigma)$.

For any split $D \in \Gamma(A, B, \Sigma)$ we also have that $\Gamma(A, D, \Sigma) \subseteq \Gamma(A, B, \Sigma)$. To see this, consider some split $C \in \Sigma$ such that $A \downarrow C$ and $C \downarrow D$; because $D \downarrow B$ we also have $C \downarrow B$ and so $C \in \Gamma(A, B, \Sigma)$. Next consider a split $C \in \Sigma$ such that $A \downarrow C$ and $C < D$; because $D < B$ we also have $C < B$ and so $C \in \Gamma(A, B, \Sigma)$. Since these two cases cover every $C \in \Gamma(A, D, \Sigma)$ we may conclude that $\Gamma(A, D, \Sigma) \subseteq \Gamma(A, B, \Sigma)$.

Now, Lemma 82 implies that the relation $A <_o B \equiv (A \downarrow B) \vee (A < B)$ is a strict partial order. Therefore, the set $\Gamma(A, B, \Sigma)$ must contain at least one minimal element D under $<_o$. That is, there must exist a split $D \in \Gamma(A, B, \Sigma)$ such that no split C in $\Gamma(A, B, \Sigma)$ has $C \downarrow D$ or $C < D$. Since $\Gamma(A, D, \Sigma) \subset \Gamma(A, B, \Sigma)$ this implies that $\Gamma(A, D, \Sigma)$ is empty. Because $\Gamma(A, D, \Sigma)$ is empty and $A \Downarrow D$ the conditions of theorem 100 are met for A and D. Therefore theorem 100 indicates that $R_{A,D}(\Sigma)$ is a multiconnected tree, and Lemma 103 shows that $|W(R_{A,D}(\Sigma))| < |W(\Sigma)|$.

We now consider a sequence of split sets $\Sigma^{(i)}$ that are obtained by starting with $\Sigma^{(0)} = \Sigma$ and then defining $\Sigma^{(i+1)} = \widehat{R}_{A,D_i}(\Sigma^{(i)})$, where D_i is any minimal element in $\Gamma(A, B, \Sigma^{(i)})$. As we have just shown, $\Sigma^{(i+1)}$ must be a multiconnected tree, and $|W(\Sigma^{(i+1)})| < |W(\Sigma^{(i)})|$. We note that $D_i \in \Gamma(A, B, \Sigma^{(i)})$ will be removed from $\Sigma^{(i+1)}$ and replaced with $R_{A,D_i}(D_i)$ and that $A < R_{A,D_i}(D_i)$. Therefore, $\Gamma(A, B, \Sigma^{(i+1)})$ will decrease by at least one element compared to $\Gamma(A, B, \Sigma^{(i)})$, since D_i is not an element of $\Gamma(A, B, \Sigma^{(i+1)})$ and $R_{A,D_i}(D_i)$ is not either. Since the size of $\Gamma(A, B, \Sigma)$ is finite, repeated application of this procedure must therefore terminate in a multiconnected tree $\Sigma^{(n)}$ for some n in which $\Gamma(A, B, \Sigma^{(n)})$ is empty and all elements $C \in \Gamma(A, B, \Sigma)$ have been replaced with $R_{A,C}(C)$. We may then apply Theorem 100 to A and B in $\Sigma^{(n)}$, obtaining a final multiconnected tree $\Sigma' = \widehat{R}_{A,B}(\Sigma^{(n)})$ in which B is replaced by $R_{A,B}(B)$. By Lemma 106, $\Sigma' = \widehat{U}_{A,\{B\}\cup\Gamma(A,B,\Sigma)} = \widehat{S}_{A,B,\Sigma}(\Sigma)$ and so $\widehat{S}_{A,B,\Sigma}(\Sigma)$ is a multiconnected tree. Furthermore, $|W(\Sigma^{(i)})|$ decreases with increasing i and so $|W(\widehat{S}_{A,B,\Sigma}(\Sigma)| < |W(\Sigma)|$. □

We note that, in order to resolve A to the right of B, the function $\widehat{S}_{A,B,\Sigma}(\cdot)$ modifes the splits that A wanders over that are *either* to the left of B *or* above B.

Lemma 111. *If any split in $B \in \Sigma$ is partial, then there exists another split $A \in \Sigma$ such that $A \Downarrow B$ and there is no element $C \in \check{\Sigma}$ such that $A \downarrow C \downarrow B$.*

Proof. Let $\mathcal{L}$ be the complete taxon set. If B is a partial split in Σ, then $\mathcal{L} - B_1 - B_2$ is not empty and there exists some $x \in \mathcal{L} - B_1 - B_2$. Then the split $D = x|\mathcal{L} - x$ satisfies $D \downarrow B$.

Now consider the set of all ordered splits $C \in \check{\Sigma}$ such that $C \downarrow B$. We have just shown that this set is not empty. Now, $<_o$ is a strict partial order on this set, and so there must be a maximal element A. This means that there can be no other split C such that $A < C$ and $C \downarrow B$. Therefore $A \Downarrow B$. Furthermore, there can be no element $C \in \check{\Sigma}$ such that $A \downarrow C \downarrow B$. $\square$

Lemma 112. *Given that A and B are ordered splits and $A \downarrow B$ then $\langle\{A, B\}\rangle = \langle\{A, R_{A,B}(B)\}\rangle \cup \left\langle\left\{A, R_{A,B^t}(B)\right\}\right\rangle$.*

Proof. If we consider a tree $\tau \in \langle\{A, B\}\rangle$, then at least one branch of τ induces a full split α that implies A, and at least one branch induces a full split β that implies B. These two branches of τ must be distinct because any branch that implied both sub-partitions would either imply $A_1 + B_1|A_2 + B_2$ or $A_1 + B_2|A_2 + B_1$. However, the left and right sides of these splits are not disjoint, since $B_1 \cup B_2 \subset A_1$. So, a single branch of the tree cannot induce a split that implies both A and B.

Because α and β are distinct full splits, they must have one of the four relationships based on "$>$". The possibility $\alpha^t < \beta$ is ruled out because it implies that $A_2 \cap B_2 = \emptyset$, which contradicts the premise $A \downarrow B$. Likewise, $\beta < \alpha$ is ruled out because it implies $B_1 \cap A_2 = \emptyset$, which also contradicts the premise $A \downarrow B$. This leaves the two cases $\alpha < \beta$ and $\alpha < \beta^t$, that correspond to the branch implying α being on alternate sides of the branch implying β. Therefore, every tree $\tau \in \langle\{A, B\}\rangle$ must have branches with splits α and β that imply A and B respectively, and for which $\alpha < \beta$ or $\alpha < \beta^t$.

If $\alpha < \beta$ then $\alpha_1 \subset \beta_1$. Since $A_1 \subseteq \alpha_1$, this means that $A_1 \subseteq \beta_1$. Therefore $\beta \implies A_1 + B_1|B_2 = R_{A,B^t}(B)$.

If $\alpha < \beta^t$ then $A_1 \subseteq \alpha_1 \subset \beta_2$. Therefore $\beta \implies B_1|B_2 + A_1 = R_{A,B}(B)$.

Therefore, every $\tau \in \langle\{A, B\}\rangle$ must also exhibit the split $R_{A,B}(B)$ or $R_{A,B^t}(B)$ and so $\langle\{A, B\}\rangle = \langle\{A, R_{A,B}(B)\}\rangle \cup \left\langle\left\{A, R_{A,B^t}(B)\right\}\right\rangle$. $\square$

Lemma 113. *Given that $A \downarrow B$ and $C < B$ then $\langle R_{A,B}(B)\rangle \cap \langle C\rangle = \langle R_{A,B}(B)\rangle \cap \langle R_{A,C}(C)\rangle$.*

Proof. Any tree $\tau \in \langle R_{A,B}(B)\rangle \cap \langle C\rangle$ must have a branch inducing a full split β that implies $B_1|A_1 + B_2$ and a branch inducing a full split γ that implies C. Since β and γ are compatible full splits, and since $C_1 \cap B_1$ and $C_2 \cap B_2$ are not empty, the possibilities for how β and γ may relate are $\gamma < \beta$ and $\beta < \gamma$. It is also possible that $\beta = \gamma$ if one branch induces both splits. We consider each case in turn.

If $\gamma < \beta$ then $A_1 + B_2 \subseteq B_2 \subset \gamma_2$. Since $C_1 \subseteq \gamma_1$ and $C_2 \subseteq \gamma_2$ this means that $\gamma \implies C_1|C_2 \cup A_1$.

If $\gamma = \beta$ then $A_1 + B_1 \subseteq \beta_2 \subseteq \gamma_2$. Therefore $\gamma_2 \implies C_1|C_2 + A_1$ as in the previous case.

If $\beta < \gamma$ then $C_2 \subseteq \beta_2$. Also $C_1 \subset B_1 \subseteq \beta_1$ because $C < B$, and $A_1 \subset \beta_2$. So $\beta \implies C_1|C_2 \cup A_1$.

Therefore, in every case, at least one of β or γ implies $R_{A,C}(C)$, and so we must have $\tau \in \langle R_{A,C}(C)\rangle$. This means that

$$\begin{aligned}\langle R_{A,B}(B)\rangle \cap \langle C\rangle &= \langle R_{A,B}(B)\rangle \cap \langle C\rangle \cap \langle R_{A,C}(C)\rangle \\ &= \langle R_{A,B}(B)\rangle \cap \langle R_{A,C}(C)\rangle\,.\end{aligned}$$

The last step follows from the fact that $\langle C\rangle \cap \langle R_{A,C}(C)\rangle = \langle R_{A,C}(C)\rangle$. $\square$

Theorem 114. *Given that Σ is a multiconnected tree and $A \Downarrow B$ in Σ and $\Theta(A, B, \Sigma)$ is empty, then $\langle\Sigma\rangle \cap \left\langle \widehat{R}_{A,B}(B)\right\rangle = \left\langle \widehat{S}_{A,B,\Sigma}(\Sigma)\right\rangle$.*

Proof. We note that if $C \in \Lambda(A,B,\Sigma)$ then $\langle C\rangle \cap \left\langle \widehat{R}_{A,B}(B)\right\rangle = \left\langle \widehat{R}_{A,C}(C)\right\rangle \cap \left\langle \widehat{R}_{A,B}(B)\right\rangle$ by Lemma 113. This allows us to write

$$\begin{aligned}
\langle\Sigma\rangle \cap \left\langle \widehat{R}_{A,B}(B)\right\rangle &= \langle\Sigma\rangle \cap \left(\bigcap_{C\in\Lambda(A,B,\Sigma)} \langle C\rangle\right) \cap \left\langle \widehat{R}_{A,B}(B)\right\rangle \\
&= \langle\Sigma\rangle \cap \left(\bigcap_{C\in\Lambda(A,B,\Sigma)} \langle C\rangle \cap \left\langle \widehat{R}_{A,B}(B)\right\rangle\right) \\
&= \langle\Sigma\rangle \cap \left(\bigcap_{C\in\Lambda(A,B,\Sigma)} \left\langle \widehat{R}_{A,C}(C)\right\rangle \cap \left\langle \widehat{R}_{A,B}(B)\right\rangle\right) \\
&= \langle\Sigma\rangle \cap \left(\bigcap_{C\in\Lambda(A,B,\Sigma)} \left\langle \widehat{R}_{A,C}(C)\right\rangle\right) \cap \left\langle \widehat{R}_{A,B}(B)\right\rangle \\
&= \langle\Sigma\rangle \cap \langle M_{A,B,\Sigma}\rangle
\end{aligned}$$

Now, let us write $M_{A,B,\Sigma} \implies C$ to mean that some member of $M_{A,B,\Sigma}$ implies C. Then $M_{A,B,\Sigma} \implies C$ allows us to conclude that $\langle C\rangle \cap \langle M_{A,B,\Sigma}\rangle = \langle M_{A,B,\Sigma}\rangle$. Therefore

$$\begin{aligned}
\langle\Sigma\rangle \cap \left\langle \widehat{R}_{A,B}(B)\right\rangle &= \langle\Sigma\rangle \cap \langle M_{A,B,\Sigma}\rangle \\
&= \left(\bigcap_{C\in\Sigma:\neg M_{A,B,\Sigma} \implies C} \langle C\rangle\right) \cap \left(\bigcap_{C\in\Sigma: M_{A,B,\Sigma} \implies C} \langle C\rangle\right) \cap \langle M_{A,B,\Sigma}\rangle \\
&= \left(\bigcap_{C\in\Sigma:\neg M_{A,B,\Sigma} \implies C} \langle C\rangle\right) \cap \langle M_{A,B,\Sigma}\rangle .
\end{aligned}$$

But $\widehat{S}_{A,B,\Sigma}(\Sigma) = \{C \in \Sigma : \neg M_{A,B,\Sigma} \implies C\} \cup M_{A,B,\Sigma}$. Therefore, $\langle\Sigma\rangle \cap \langle R_{A,B}(B)\rangle = \left\langle \widehat{S}_{A,B,\Sigma}(\Sigma)\right\rangle$. □

Theorem 115. *Given that Σ is a multiconnected tree and $A \Downarrow B$ in Σ and $\Theta(A,B,\Sigma)$ is empty, then $\langle\Sigma\rangle = \left\langle \widehat{S}_{A,B,\Sigma}(\Sigma)\right\rangle \cup \left\langle \widehat{S}_{A,B^t,\Sigma}(\Sigma)\right\rangle$.*

Proof. We first note that by Lemma 112, $\langle A\rangle \cap \langle B\rangle = \langle A\rangle \cap \left(\langle R_{A,B}(B)\rangle \cup \langle R_{A,B^t}(B)\rangle\right)$. We then express Σ as follows, and apply this fact:

$$\begin{aligned}
\langle\Sigma\rangle &= \langle\Sigma\rangle \cap (\langle A\rangle \cap \langle B\rangle) \\
&= \langle\Sigma\rangle \cap \langle A\rangle \cap \left(\left\langle \widehat{R}_{A,B}(B)\right\rangle \cup \left\langle \widehat{R}_{A,B^t}(B)\right\rangle\right) \\
&= \langle\Sigma\rangle \cap \left(\left\langle \widehat{R}_{A,B}(B)\right\rangle \cup \left\langle \widehat{R}_{A,B^t}(B)\right\rangle\right) \\
&= \langle\Sigma\rangle \cap \left(\left\langle \widehat{R}_{A,B}(B)\right\rangle \cup \left\langle \widehat{R}_{A,B^t}(B^t)\right\rangle\right) \\
&= \left[\langle\Sigma\rangle \cap \left\langle \widehat{R}_{A,B}(B)\right\rangle\right] \cup \left[\langle\Sigma\rangle \cap \left\langle \widehat{R}_{A,B^t}(B^t)\right\rangle\right] .
\end{aligned}$$

Now, by theorem 114 this implies that $\langle\Sigma\rangle = \left\langle \widehat{S}_{A,B,\Sigma}(\Sigma)\right\rangle \cup \left\langle \widehat{S}_{A,B^t,\Sigma}(\Sigma)\right\rangle$ and we are done. □

Lemma 116. *Given that Σ is a multiconnected tree and $A \Downarrow B$ in Σ and $\Theta(A,B,\Sigma)$ is empty then quartets $q(\widehat{S}_{A,B}(\Sigma)) \cap q(\Sigma)^C$ are all of the form $c_1c_1|a_1c_2$ for some $C \in \Lambda(A,B,\Sigma) \cup \{B\}$.*

Now, the quartets in $\widehat{S}_{A,B,\Sigma}(\Sigma)$ that are not in Σ may be written as follows, since $\Gamma(A,B,\Sigma) = \Lambda(A,B,\Sigma)$:

$$\begin{aligned}
q(\widehat{S}_{A,B}(\Sigma)) \cap q(\Sigma)^C &= q\left(\{C \in \Sigma : \neg M_{A,B,\Sigma} \implies C\} \cup M_{A,B,\Sigma}\right) \cap q(\Sigma)^C \\
&= \left[q\left(\{C \in \Sigma : \neg M_{A,B,\Sigma} \implies C\}\right) \cap q(\Sigma)^C\right] \cap \left[q\left(M_{A,B,\Sigma}\right) \cap q(\Sigma)^C\right] \\
&= q\left(M_{A,B,\Sigma}\right) \cap q(\Sigma)^C \\
&= q\left(\bigcup_{C \in \Gamma(A,B,\Sigma) \cup \{B\}} \widehat{R}_{A,C}(C)\right) \cap q(\Sigma)^C \\
&= \bigcup_{C \in \Lambda(A,B,\Sigma) \cup \{B\}} q(\widehat{R}_{A,C}(C)) \cap q(\Sigma)^C
\end{aligned}$$

Now $\hat{R}_{A,C,}(C) = C_1|C_2 + A_1$ so the quartets $q(\widehat{R}_{A,C}(C))$ will be of the form $c_1c_1|c_2c_2$ or $c_1c_1|a_1c_2$ or $c_1c_1|a_1a_1$. However, A already implies quartets of the form $c_1c_1|a_1a_1$ since $A \downarrow C$. Also, C implies quartets of the form $c_1c_1|c_2c_2$. Thus the intersection with $q(\Sigma)^C$ will remove quartets of the first and third forms, leaving only quartets of the form $c_1c_1|a_1c_2$.

Lemma 117. *Given that Σ is a multiconnected tree and $A \Downarrow B$ in Σ and $\Theta(A,B,\Sigma)$ is empty then quartets $q(\widehat{S}_{A,B}(\Sigma)) \cap q(\Sigma)^C$ and the quartets $q(\widehat{S}_{A,B^t}(\Sigma)) \cap q(\Sigma)^C$ do not intersect.*

Proof. Quartets $q(\widehat{S}_{A,B}(\Sigma)) \cap q(\Sigma)^C$ are of the form $c_1c_1|a_1c_2$ where $C \in \Lambda(A,B,\Sigma) \cup \{B\}$ by Lemma 116. Now, each such C has either $C_1 = B_1$ or $C_1 \subset B_1$, so the quartets are of the form $b_1b_1|a_1c_2$. Note that c_2 cannot be in A_1 since C_2 and A_1 are disjoint.

Quartets $q(\widehat{S}_{A,B^t}(\Sigma)) \cap q(\Sigma)^C$ are of the form $c_1c_1|a_1c_2$ where $C \in \Lambda(A,B^t,\Sigma) \cup \{B^t\}$, and for each such C, we have $C_1 \subset B_2$. Therefore these quartets are of the form $b_2b_2|a_1c_2$. Note that c_2 cannot be in A_1 since C_2 and A_1 are disjoint.

However, B_1 and B_2 are non-overlapping, and A_1 is disjoint from both B and from all the other splits C. Therefore no quartet of the form $b_1b_1|a_1c_2$ that divides a single element of A_1 from two elements of B_1 can also divide a single element of A_1 from two elements of B_2. Therefore the two sets of quartets are disjoint. □

Closure We seek to show that if Σ is a set of partial splits that satisfies the criteria for being a multiconnected then the set of quartets $q(\Sigma)$ implied by all the individual splits is exactly the same as the set of quartets $q\langle\Sigma\rangle$ common to all trees that are compatible with Σ. That is, Σ does not jointly imply any quartets that are not implied by individual splits $\sigma \in \Sigma$. We reduce closure of multiconnected trees to closure of (possibly multifurcating) trees.

Compatibility We also reduce compatibility of multiconnected trees to compatibility of (possibly multifurcating) trees.

Theorem 118. *If a set of splits Σ is a multiconnected tree then $\langle\Sigma\rangle$ is not empty (Σ is compatible) and for any $x \notin q(\Sigma)$ then $x \notin q(\langle\Sigma\rangle)$.*

Proof. We define a sequence of split sets $\Sigma^{(i)}$ starting with $\Sigma^{(0)} = \Sigma$. Now, if $\Sigma^{(i)}$ is a multiconnected tree with $|W(\Sigma^{(i)})| > 0$ then by Lemma 111 we can find a pair of splits $A_i, B_i \in \Sigma^{(i)}$ such that

$A_i \Downarrow B_i$ in $\Sigma^{(i)}$. If $x \notin q(\Sigma^{(i)})$ then by Lemma 117 one or both of $S_{A_i,B_i,\Sigma(i)}(\Sigma^{(i)})$ and $S_{A_i,B_i^t,\Sigma(i)}(\Sigma^{(i)})$ does not imply x. We therefore set $\Sigma^{(i+1)} = S_{A_i,B_i,\Sigma^{(i)}}(\Sigma^{(i)})$ unless $x \in q\left(S_{A_i,B_i,\Sigma(i)}(\Sigma^{(i)})\right)$, in which case we set $\Sigma^{(i+1)} = S_{A_i,B_i^t,\Sigma^{(i)}}(\Sigma^{(i)})$. By theorem 110 each $\Sigma^{(i+1)}$ is a multiconnected tree if $\Sigma^{(i)}$ is one; furthermore $|W(\Sigma^{(i+1)})|$ must be smaller than $|W(\Sigma^{(i)})|$ by theorem 110. $\left\langle\Sigma^{(i+1)}\right\rangle \subseteq \left\langle\Sigma^{(i)}\right\rangle$ by theorem 115. Since $|W(\cdot)|$ may be at most zero, and since it decreases at each step where it is greater than zero, the sequence $\left|W(\Sigma^{(i)})\right|$ must eventually terminate at some $i = n$ where $\left|W(\Sigma^{(n)})\right| = 0$ and $x \notin q(\Sigma^{(n)})$. Since each entry in this $\Sigma^{(n)}$ is a full split and all the full splits are pairwise compatible because $\Sigma^{(n)}$ is a multiconnected tree, $\Sigma^{(n)}$ must be jointly compatible, so that $\left\langle\Sigma^{(n)}\right\rangle$ is not empty. Since $\left\langle\Sigma^{(n)}\right\rangle \subseteq \langle\Sigma\rangle$ this means that Σ is also compatible. Furthermore, $\left\langle\Sigma^{(n)}\right\rangle \subseteq \langle\Sigma\rangle$ implies that $q(\langle\Sigma\rangle) \subseteq q\left(\left\langle\Sigma^{(n)}\right\rangle\right)$. By theorem **??** $q\left(\left\langle\Sigma^{(n)}\right\rangle\right) = q(\Sigma^{(n)})$ and so $q(\langle\Sigma\rangle) \subseteq q(\Sigma^{(n)})$. Since $x \notin q(\Sigma^{(n)})$ then $x \notin q(\langle\Sigma\rangle)$. □

A.4

B

C

D

E Examples/Corner Cases

E.1 X is separated from a -X branch

12X|345

123|45

E.2 Two branches wander over each over

X=12|34

Y = 12|45

E.3 A branch wanders directly that did not before, because an intermediate branch goes away.

A=12|34

B=1234|X56

C=1234Y|56

$B < C \downarrow A$ and so $B \downarrow A$.

$X \downarrow C$

$Y \downarrow C \downarrow A$

We resolve $X \downarrow C$ to $X < C'^t$ where

$C' = 1234Y|X56$

This makes B go away.

Now C' wanders directly over A whereas previously we had $B < C \downarrow A$.

E.4 Illegal

Here, if 1 really wanders over B, then it can attach on the 56 side of B, when B is on the 34 size of A. This violates the split A.

A=12|34

B=234|56

E.4.1 Illegal

Here, if 2 really wanders over B, then 2 can attach on the 56 side of B when B is on the 34 side of A, thus violating the split A.

A=12|34

B=13|56

E.4.2 Illegal #(A,B)=2

A=12 | 45

B=23|56

If $A < B$ then we have 12|456 < 123|56

If $B < A$ then we have 23|456 < 123|45

Thought: "The problem that we have with these two ***splits*** is that they are really the same ***branch***."

E.4.3 Illegal #(A,B)=2 / 2-closure.

A=12X | 45Y

B=2X3|5Y6

If $A < B$ then we have 12X|45Y6 < 12X3|5Y6

If $B < A$ then we have 2X3|45Y6 < 12X3|45Y

Thought: "The problem that we have with these two ***splits*** is that they are really the same ***branch***."

Also, we get 2X|45Y6 and 12X3|5Y. In short, we get either $A_1 \cup B_1|A_2 < B_1|A_2 \cup B_2$ or $A_1 \cup B_1|B_2 < A_1|A_2 \cup B_2$, and therefore we *always* get $A_1 \cup B_1|A_2 \cap B_2$ and $A_1 \cap B_1|A_2 \cup B_2$.

E.4.4 Illegal: 6 branches too similar to each other

23|456

12|456

13|456

123|56

123|46

123|45

All of these splits are almost identical to the full split 123|456 except for the lack of a single leaf taxon.

We must pick an order for

F Splitting out part of the paper?

Can I split out part of the paper that (a) identifies partial splits as what we should be concerned about and (b) proposes marked branches as ways of displaying them, as well as (c) multiconnected trees and cloud diagrams? I can show (d) that not all split sets can be represented as either (b) or (c).

This would involve defining how a multiconnected graph implies a split: an edge (u, v) implies a split A if $x \in A_1$ is connected to u but not v in every embedded graph minus (u, v), the same is true for $y \in A_2$, and every other $z \in L - A_1 - A_2$ is connected to u in some embedded graphs and v in others.

I could then use illustrative examples to illustrate the importance of partial splits without necessarily proposing a general alternative. This would mean that, when such an alternative exists, we would be able to draw it because we would know what it means.

F.1 Can I use a search method instead of a proof?

Can I *search* for multiconnected trees? I could do this by (theoretically) generating every embedded tree and checking the result...

F.2 The algorithm for finding the partial splits

I could (optionally) decide to include the algorithm to find the partial splits, and the graph of supported partial splits versus supported full splits.